Ns G-99-60

255 p.

N64-19941

CODE-1

NASA CR-53870

DEI SUB NUMINE VIGET

OTS PRICE

XEROX $ 16.50 ph

MICROFILM $ 2.85 mf.

PRINCETON UNIVERSITY

DEPARTMENT OF
AEROSPACE AND MECHANICAL SCIENCES

RC #1

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

NASA GRANT NsG-99-60

A THEORETICAL STUDY OF NON-LINEAR COMBUSTION INSTABILITY: LONGITUDINAL MODE

Technical Report No. 677

Department of Aerospace and Mechanical Sciences

UNPUBLISHED PRELIMINARY DATA

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March 1964

ABSTRACT

Periodic solutions of finite amplitude are found for two different models of nonlinear longitudinal wave-type oscillations in a rocket combustor. In the first model, the characteristic time of the combustion process is negligible compared to the wave travel time in the chamber, or equivalently, the period of oscillation. Here, no phase or time-lag exists between energy addition from combustion and pressure. This model is believed to apply to premixed gas rockets where chemical kinetics seem to provide the forcing function. In the second model, the wave travel time and the characteristic time of the combustion process are of the same order of magnitude, so that a phase can exist between energy addition and pressure. Specifically, the Crocco time-lag concept is employed to introduce the characteristic time in this model. This concept has proven successful in predicting the stability behavior of liquid rockets when small perturbations occur and is extended to the nonlinear case in this work. In both models the chamber is considered to be of sufficient length to allow the approximations of concentrated combustion at the injector end of the chamber and short nozzle at the other end.

The theory predicts that unstable operation is possible for both cases within a certain range of the parameters which describe the feedback from the combustion process to the wave phenomenon. The periodic solutions may be stable or

unstable to small disturbances of the amplitude. It is found that there are solutions in the range of practical interest, with respect to liquid rockets, which are unstable. This is shown to mean that "triggering" of an oscillation by a finite disturbance may occur in the practical range of operation. The only stable solutions should contain shock waves. An important relationship exists between the forcing function of the oscillation and the wave form. This indicates the possibility that something could be learned about the combustion process by observing the wave form in the chamber. As the amplitude goes to zero the results are identical with the results of small perturbation analyses.

Preliminary data indicate qualitative agreement between theory and experiment on both a premixed gas rocket (for the first model) and a liquid rocket (for the second model).

TABLE OF CONTENTS

	Page
Title Page	i
Abstract	ii
Table of Contents	iv
Nomenclature	v
Chapter I: INTRODUCTION	1
Nature of the Physical Problem	1
Approach to the Theoretical Problem	4
Theoretical Models	9
Mathematical Technique	11
Chapter II: A SHOCK WAVE MODEL OF UNSTABLE ROCKET COMBUSTORS	14
Analysis	14
Numerical Example	32
Chapter III: NONLINEAR COMBUSTION INSTABILITY WITH TIME-LAG EFFECTS: LONGITUDINAL MODE	34
Formulation of the Problem	34
Solution of the Equations	42
First Order Solution	63
Second Order Solution	72
Third Order Solution	88
Stability of the Periodic Solutions	125
Wave Form of the Oscillations	135
Numerical Example	143
Chapter IV: DISCUSSION OF RESULTS	151
Analytical Results	151
Extension of the Analytical Work	166
Comparison with Experiments	169
Conclusions	173
References	175
Figures	
Appendix A: Nonlinear Mechanical Analogy	
Appendix B: Investigation of Combustion Zone Dynamics	
Appendix C: Small Perturbation Analysis	
Appendix D: Investigation of Solutions to Differential Equation of Chapter II	
Appendix E: Nonlinear Extension of Crocco Time-lag Postulate	

NOMENCLATURE

General:

x	=	longitudinal space dimension
t	=	time
u	=	gas velocity
a	=	speed of sound
p	=	pressure
ρ	=	density
T	=	temperature
α, β	=	characteristic coordinates
$P/2, Q/2$	=	Riemann invariants
F, G	=	homogeneous solutions for x and t
R	=	inhomogeneous part of boundary condition at combustion zone
ϵ	=	perturbation parameter
γ	=	ratio of specific heats
ν	=	$\frac{\gamma-1}{2} u_o$
$\dot{m}$	=	mass flow per unit area per unit time

Subscript: $i = 0, 1\ 2$, etc. indicates coefficients of i^{th} term series expansion in parameter ϵ.

Chapter II and Appendices B and C:

λ	=	thermal conductivity
ε_h	=	turbulent exchange coefficient for heat transfer
r	=	energy release rate per unit volume
$\mathcal{T}$	=	period of oscillation
$\varepsilon\Delta\alpha$	=	"slippage" of characteristics
ω, δ	=	parameters related to the combustion process
V	=	shock velocity

Subscript: AB, BC, CD indicates direction of shock wave (see Figure 3)

Chapter III and Appendix E:

ψ_α, ψ_β = scaled characteristic coordinates

τ = time-lag

b = time-lag in characteristic coordinates

ϕ = time-lag in scaled characteristic coordinates

n = interaction index

N,M,L = coefficients in Taylor series expansion for rate function of combustion process

r = exponential growth or decay factor

s = frequency in characteristic coordinates

ω = frequency

k = $\dfrac{1+\nu}{1-\nu} = \dfrac{1+\frac{\gamma-1}{2}u_o}{1-\frac{\gamma-1}{2}u_o}$

A = amplitude coefficient for second harmonic term in the wave form

B = amplitude coefficient for the third harmonic term in the wave form

C = coefficient for correction in mean flow properties

θ = phase factor between first and second harmonics

Δ = phase factor between first and third harmonics

D = coefficient of displacement from the neutral line

λ_1, λ_2 = functions related to P,Q,F and G

Note: k_1, k_1^*, r_1 through r_4, V_1, V_1^*, V_2, m_1, c_1, d_1, g_1, h_1 through m_{21}, c_{21}, d_{21}, g_{21}, h_{21}, Z_1 through Z_6, ℓ_r, ℓ_i, b_r, b_i, ξ_r, ξ_i, B_r, B_1, ω_1, ω_2, ω_3, y_1, y_2, y_3, a_r, a_i, c_r, η_r, η_i are all constants defined in Chapter III.

Subscript: i = 0,1,2, etc. indicates coefficients in series expansion in ϵ of unsteady quantities about their steady-state values.

Superscript: i = 0,1,2, etc. indicates coefficients in expansion in ϵ of characteristic parameters ("eigenvalues") of solution about their values for neutral oscillation with infinitesimal amplitudes.

CHAPTER I

INTRODUCTION

NATURE OF THE PHYSICAL PROBLEM

High frequency combustion instability involves wave-type oscillations of pressure and velocity within the combustion chamber and exhaust nozzle of the rocket motor. Energy is supplied to maintain the oscillation by the combustion process. The occurrence of instability in actual rocket motors is a serious problem, since large heat transfer rates to the walls are associated with the high amplitude waves. This often results in the "burn-out" of the motor, near the injector or at nozzle section. This instability phenomenon has been observed in both liquid and solid propellant rockets as well as in experimental pre-mixed gas rockets.

The instability may occur in the longitudinal, transverse, or mixed mode, depending mainly on the geometry of the chamber. The waveform of the longitudinal mode usually consists of shock waves followed by exponential decays with time* in pressure and velocity. Sinusoidal waveforms have been observed, however, particularly near the stability limits. Continuous waveforms with steep "peaks" and shallow "valleys" have been observed for the transverse mode in full cylindrical chambers. Shocks or detonations have been observed for this mode in annular chambers. Figure 1 shows pressure waveforms of various types.

Crocco, Grey, Harrje, and Reardon have observed upper

* The waveform is considered with variable time and fixed position.

and lower-length stability limits as well as off-resonant oscillation for the longitudinal mode in liquid rockets using variable-length chambers. They have observed corresponding effects for the transverse mode. These experiments have provided evidence of the existence of an important characteristic time associated with the forcing function of the instability. They have also produced results which are in good quantitative agreement with the predictions of the Crocco time-lag theory. (See Ref. 1, 2, 3, and 4.) Of course, that theory involves a linearized treatment and applies only at the stability limit. However, on the basis of its success, it is reasonable to expect that the time-lag effect can be equally important away from the stability limit. In that region away from the limit, finite-amplitude waves may exist. It is noteworthy that in the experimental verification of the time-lag theory, the frequencies of finite-amplitude waves were measured in motors operating very close to the stability limit but not at the limit.

The instability may be developed in two different manners; by a spontaneous action (occurring within the combustion chamber) or by a "triggering" action (from inside or outside of the chamber). The former type arises from small perturbations in the flow field. These perturbations are forced to grow by energy addition from the combustion process. The latter type is caused by large disturbances to the system, such as, abrupt changes in the feed system operation. In the case of many experimental motors, artificial pulses are used (generated by gun powder blasts or high-pressure gas blasts). The onset of the spontaneous instability may, in principle, be treated by a linearized analysis. However, the onset of the "triggered" instability is nonlinear in nature and cannot be treated by a linearized analysis. Both types of instability will involve nonlinear effects. These effects are present with any finite-amplitude wave so that they are important whenever a small perturbation grows in amplitude. In the longitudinal mode, the nonlinear effects can be important even for small amplitudes, since wave distortion and shock formation may occur. Both types of instability, although developed differently, result in similar cyclic oscillations when fully established.

References 5 and 6 discuss experiments involving pulsing techniques for both the longitudinal and transverse modes in liquid rockets. References 7 and 8 discuss pulsing techniques for the longitudinal mode in solid rockets. It has been clearly demonstrated that high frequency instability may be triggered under certain operating conditions when sufficiently energetic pulses are applied. Figure 2 (which is borrowed from Ref. 5) shows an example of this "triggered" instability in a graphic manner. The results are for a liquid rocket designed to allow only transverse oscillation. A variable-grain gun powder charge produces the pulse.

The important physical aspects of the problem are that combustion instability occurs with different waveforms for different modes and that off-resonant oscillation and "triggering" action are possible.

APPROACH TO THE THEORETICAL PROBLEM

A common approach to the nonlinear instability problem involves the numerical integration of the equations of motion (rewritten in some convenient form) based on certain initial conditions. These initial conditions must be that some wave or disturbance exists in the combustion chamber at the initial time. The disturbance may have any one of an infinite number of functional forms. The numerical integration of the equations determines whether this disturbance decays in amplitude (described as "stable") or grows in amplitude ("unstable"). However, only the stability of the particular disturbance taken initially will be determined. The stability of the motor can be determined only if all possible initial disturbances are examined and if the stability of each is determined. Therefore, this approach has very grave shortcomings.

The present work uses a different approach. Since combustion instability is observed to be a cyclic phenomenon when fully established, a theory may be developed which uses the cyclic condition rather than initial conditions. The periodic solution which results from this approach may be either stable or unstable; i.e., any small perturbations to the amplitude associated with the periodic solution may either grow (unstable) or decay (stable). If the solution is stable, the amplitude of the fully established oscillation has been found. On the other hand, if the solution is unstable, the amplitude of the disturbance sufficient to "trigger" the cyclic oscillation has been found. This type of approach has often been used in the analysis of nonlinear oscillations but has never before been applied to the combustion instability problem.

Any nonlinear theory developed for liquid rockets should include a time-lag* effect. Until the driving mechanism of the instability is understood, the time-lag concept cannot be formulated analytically in a proper manner. The heuristic formulation presented by Crocco (Ref. 1) appears to be a reasonable choice at this time. A theory applicable to other types of rockets, such as a premixed gas rocket, where phasing effects are not important, should not contain a time-lag effect. It should be noted that a theory without time-lag effects may be considered as a limiting case for liquid rockets.

Before attacking the nonlinear combustion instability problem, it is useful to examine three topics; 1) nonlinear mechanical analogies, 2) phenomena closely related to the combustion instability phenomenon, and 3) the various gas-dynamical processes involved in rocket instability. Appendix A contains the analysis of a nonlinear ordinary differential equation with time-lag effects. This shows that off-resonant oscillations are possible due to the time-lag effect. The resonant solution is obtained in the limiting case of no time-lag. It is seen that the initial conditions may be replaced by the

*The feedback of energy from the combustion process to the chamber oscillation at any instant depends upon the thermodynamic conditions at the combustion zone over a finite period of time.

cyclic condition. The handling of the perturbation technique (including eigenvalue perturbations) for nonlinear equations is demonstrated.

The most interesting works on phenomena closely related to nonlinear combustion instability were performed by Maslen and Moore (Ref. 9 and 10) and Chu (Ref. 11 and 12). Maslen and Moore analyzed the case of irrotational transverse waves in a cylindrical chamber filled with gas at zero mean flow. They found a periodic solution with a nonlinear waveform which consisted of high, sharp "peaks" and shallow, long "valleys". These waveforms are similar to those found in actual rocket combustion chambers. Their chamber is different from an actual combustion chamber in that the effects of mass and energy addition are neglected, and that there is no nozzle. All these effects are of the order of the mean flow or of some mean steady-state Mach number in the chamber. When applied to an actual chamber, the error in these should be of the order of the Mach number. This is often small and explains the similarity in the waveforms. A standard perturbation procedure was used including an eigenvalue (frequency) perturbation. Since the frequency perturbation was of second order, a third order analysis was necessary.

Chu (Ref. 11 and 12) analyzed the case of resonant, longitudinal oscillations in a one-dimensional chamber filled with gas at zero mean flow. Energy, but no mass, was added at one end of the chamber while a solid wall stood at the other end. The cyclic condition replaced the initial conditions. As is usual for resonant oscillations, a second order analysis yielded a first order approximation to the solution. "Sawtooth" waveforms in the chamber were predicted. That is, shock waves followed by linear pressure decay (with time) were shown to exist.

Finally, some gasdynamical processes are considered which can be important in combustion chamber wave phenomena: amplitude increase, amplitude decrease, wave steepening, wave broadening, and wave reflection. Note that these need not be distinct processes.

The amplitude increases whenever energy is added with the proper phasing. The phasing depends on the characteristic time of that portion of the combustion process which involves the feedback of energy to the oscillation. That is, it depends on the duration of the pressure-sensitive portion of the combustion process. When this characteristic time and the period of oscillation are of the same order of magnitude, a coupling between the combustion process and the wave phenomenon may occur. If the coupling is sufficiently strong, unstable operation of the motor results. Instability may also result if the characteristic time of the combustion process is negligibly small compared to the period of oscillation. The energy addition and, therefore, the resulting amplitude increase are of the order of magnitude of the mean flow.

Friction, heat transfer, and shock waves are dissipative processes which result in amplitude decrease. Friction and heat transfer occur mainly near the chamber walls and have negligible effects elsewhere. Since shock wave dissipation is of third order in amplitude, it is important only for large amplitudes. The mean flow through the rocket motor convects some of the energy of oscillation through the nozzle, resulting in a decrease in amplitude. The amplitude decrease due to reflection at the nozzle end is, therefore, of the order of the mean flow.

Any compression wave moving through a uniform medium in one direction inherently tends to steepen. In the same situation, an expansion wave tends to broaden. Wave steepening would result in shock wave formation after some time. This would be followed by dissipation of the wave with eventual disappearance of the wave unless some other influence were present. For oscillations in the transverse mode in cylindical chambers, periodic solutions without shocks are possible. Here, shock formation is prevented by the "turning" effect at the outer wall. Maslen and Moore have shown that cyclic solutions are allowed even in the absence of a driving mechanism (such as combustion). This is generally not true with the longitudinal mode where cyclic waveforms usually involve shocks. In that case, the waves will dissipate unless a driving mechanism is provided.

If a wave moves through a gradually decreasing cross-sectional area, a gradual reflection of the wave occurs over a period of time. Of course, the wave is broadened in reflection. Therefore, a long nozzle in a combustion chamber inhibits shock formation.

It is also possible that wave steepening will not occur when the chamber is oscillating at off-resonant frequencies. If a "bundle" of characteristics initially representing a compression wave is followed through the space versus time plot, it is seen that the strength of the compression wave could change. In fact, the wave could even become an expansion. This results from the difference between the wave travel time and the period of flow oscillation. Therefore, in some cases, it might be expected that a longitudinal instability could exist without shock waves.

Wave reflection occurs in the longitudinal mode at both ends of the chamber. The reflection at the nozzle occurs with a loss of amplitude due to mean flow convection and with a wave broadening due to changing area. In the limit of a zero-length nozzle, no wave broadening occurs, but amplitude will decrease discontinuously on reflection. In the limit of concentrated combustion at the other end of the chamber, the amplitude increases discontinuously in reflection there. Of course, in the transverse mode, the "turning" effect previously mentioned is a reflection process.

THEORETICAL MODELS

This work deals with instability in the longitudinal mode. Two separate models are analyzed. The first involves a shock wave instability with no time-lag effects present in combustion. That is, the characteristic time of combustion is negligibly small compared to the period of oscillation. The second model assumes no shocks but contains time-lag effects. In particular, the Crocco time-lag postulate is used. In both cases, only periodic solutions are sought in some small range near the neutral stability line. The stability of the solutions must be determined, i.e., the solutions may be either stable limit cycles or unstable limit cycles.

In order to simplify the gasdynamical processes, certain assumptions are made. It is assumed that the flow is one-dimensional and the chamber cross-sectional area is constant. The chamber is assumed to be very long so that the limiting cases of zero length nozzle and concentrated combustion at the chamber end are investigated. Further, no dissipative processes are allowed except for the presence of the shock wave in the first model.

The first analysis, presented in Chapter II, yields the first order approximation to a periodic solution describing the waveform of the fundamental longitudinal mode. Here, a second order analysis yields a first order solution. A third order analysis cannot be performed since shock wave dissipation is of that order. The waveform consists of shock discontinuities followed by exponential decays in pressure and velocity. The second analysis, presented in Chapter III, yields an approximation, correct to third order, to a periodic solution without shock waves. This solution describes the waveform of any longitudinal mode when time-lag exists. The analysis is carried to third order so that the eigen value perturbations, which are of second order, could be determined. For each theoretical model, the solution explicitly relates the waveform and the wave amplitude to the parameters which characterize the combustion process. These parameters are related to the means of energy release and feedback; allowing, therefore, the possibility of investigating the driving mechanism of the instability by observing the waveform in the combustion chamber.

MATHEMATICAL TECHNIQUE

For the analysis of nonlinear wave phenomena in both one-dimensional unsteady flow and two-dimensional steady supersonic flow, it is convenient to use the characteristic coordinate perturbation technique. This method was first used to obtain higher order corrections to linearized theory for a steady two-dimensional uniform supersonic flow around a thin airfoil. (Ref. 16). Linearized airfoil theory predicts that the solution is constant along each member of one family of characteristics. However, the characteristic curves are given to zero order only and, therefore, are straight, parallel lines. Actually, they are not straight, parallel lines when nonlinear terms are considered.

If it is assumed that the characteristics are straight, parallel lines, the error accumulates as the distance from the airfoil increases. In the case of one-dimensional, unsteady flow, serious errors will result unless account is taken of the deviation of the characteristics from straight, parallel lines. In fact, in the combustion instability problem, the error can be more serious since both families of characteristics have important interactions causing curvature effects to appear to first order, while with flow around an airfoil, curvature of the characteristics is of higher order.

The technique used involves perturbing the time and space dimensions as well as the flow properties in some amplitude parameter. The characteristics of the field form the new coordinate system, so that x and t are now dependent variables. The purpose of the perturbation scheme is to obtain an infinite set of linear equations from a finite set of non-linear equations. This particular perturbation scheme yields a greater portion of homogeneous equations than an ordinary scheme yields. This scheme, therefore, simplified the analysis, although it is more abstract than the usual perturbation scheme.

The technique was developed by Lighthill (Ref. 13) and Whitham (Ref. 16) and was extended to the perturbation of both characteristic coordinates by Lin (Ref. 14) and Fox (Ref. 15). It was first applied to the problem of oscillations with shocks by Chu (Ref. 11 and 12). Those oscillations occurred in a one-dimensional chamber with zero mean flow. However, when attacking more complex problems such as the combustion instability problem, simplifications to the technique, which allow a clearer insight to the physical nature of the phenomenon, are very desirable. The present analysis introduces simplifications to the technique which involve the continuation of the solution across shock waves.

CHAPTER II

A SHOCK WAVE MODEL OF UNSTABLE ROCKET COMBUSTORS: ANALYSIS:

A shock wave model of the fundamental mode of combustion instability is investigated. The assumption is made that the characteristic time of the combustion process is negligible compared to the wave travel time. Other assumptions are:

(1) The flow is one-dimensional.

(2) The chamber cross-sectional area is constant.

(3) The chamber is very long so that the configuration may be approximated by the limiting case of zero-length nozzle and concentrated combustion zone at the chamber end.

(4) A shock wave moves back and forth the length of the chamber with a constant period, reflecting alternately from the nozzle end and the combustion end.

(5) Flow is homentropic outside of the combustion zone up to and including second order in the wave amplitude. This allows shock waves to occur but no entropy waves are allowed.

(6) The chamber gas is calorically perfect.

If the length of the combustion zone is small compared to the chamber length (as assumed above), unstable operation may be considered forced in a piston-like manner by the combustion process. The power per unit area (pressure times gas velocity) at the end of the combustion zone is a rate of energy input to the oscillation of the chamber gases.

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This power term is related to the rate of energy release within the combustion zone and, therefore, is, in general, a function of thermodynamic conditions within the zone. If appropriate space-wise mean values are used for the thermodynamic properties (only within the combustion zone) and, further, if all thermodynamic properties are related to the speed of sound, a relation* applicable at the end of the combustion zone is obtained:

$$\frac{u-u_0}{u_0} = \omega\left(\frac{a-a_0}{a_0}\right) + \delta\left(\frac{a-a_0}{a_0}\right)^2 + \begin{matrix}\text{terms}\\ \text{of}\\ \text{order}\end{matrix}\left(\frac{a-a_0}{a_0}\right)^3 \qquad (1)$$

The constants ω and δ may be calculated based on a knowledge of the combustion process. In Appendix B they are determined for a one-dimensional combustion zone where perturbations due to chemical effects are important but perturbations due to diffusion effects are negligible. The above relation is the boundary condition at one chamber end when the limiting case of concentrated combustion is considered.

All thermodynamic variables are nondimensionalized with respect to their steady-state values. The gas velocity is nondimensionalized with respect to the steady-state speed of sound, space dimension with respect to chamber length and time dimension with respect to chamber length divided by speed of sound.

The well-known compatibility relations may be obtained from the equations of unsteady, one-dimensional motion for a fluid. Under our assumptions these relations (to second order) are

*Zero subscripts denote steady-state values.

$$\frac{2}{\gamma-1}\, da + du = 0 \qquad \text{along} \qquad \frac{dx}{dt} = u + a$$

$$\frac{2}{\gamma-1}\, da - du = 0 \qquad \text{along} \qquad \frac{dx}{dt} = u - a$$

Note that integration of $\frac{dx}{dt}$ gives the two families of characteristic lines. Now let each member of the family with negative slope $\left(\frac{dx}{dt} = u-a\right)$ be characterized by a certain value of the parameter α and each member of the family with positive slope $\left(\frac{dx}{dt} = u+a\right)$ be characterized by a certain value of the parameter β. Let α and β become the new independent variables and it is said

$$u = u\,(\alpha,\beta)$$

$$a = a\,(\alpha,\beta)$$

$$x = x\,(\alpha,\beta)$$

$$t = t\,(\alpha,\beta)$$

The compatibility relations are now partial differential equations in α, β coordinates

$$\frac{2}{\gamma-1}\,\frac{\partial a}{\partial \alpha} + \frac{\partial u}{\partial \alpha} = 0$$

$$\frac{2}{\gamma-1}\,\frac{\partial a}{\partial \beta} - \frac{\partial u}{\partial \beta} = 0$$

$$\frac{\partial x}{\partial \alpha} = (u+a)\,\frac{\partial t}{\partial \alpha}$$

$$\frac{\partial x}{\partial \beta} = (u-a)\,\frac{\partial t}{\partial \beta} \qquad (2)$$

It is necessary to have a numbering system for the α, β coordinates. A convenient numbering system is shown in Figure 3. Primes are used for Region II and double primes are used for Region III. The point $t = 0$ is chosen as the time of the shock reflection at the combustion end. Number α by the value of x at its intersection with the shock wave AB. Number β by the value of α at their intersection at the $x = 0$ position (injector). Let $\beta' = \beta$ at their intersection at the shock BC. Let $\alpha' = 1 + \beta'$ at their intersection at the $x = 1$ position (nozzle). Number α'' by the value of x at its intersection with the shock CD and let $\beta'' = \alpha''$ at their intersection at $x = 0$. The cyclic condition will be implied by stating conditions along the shock CD are identical to conditions along the shock AB. Therefore, the characteristics in the other regions need not be numbered since the solutions will be cyclic.

The dependent variables are perturbed in a regular series in ε which is some amplitude parameter as yet not specifically defined

$$
\begin{aligned}
u &= u_o + \varepsilon u_1 (\alpha,\beta) + \varepsilon^2 u_2 (\alpha,\beta) + \cdots \\
a &= 1 + \varepsilon a_1 (\alpha,\beta) + \varepsilon^2 a_2 (\alpha,\beta) + \cdots \\
p &= 1 + \varepsilon p_1 (\alpha,\beta) + \varepsilon^2 p_2 (\alpha,\beta) + \cdots \\
x &= x_o (\alpha,\beta) + \varepsilon x_1 (\alpha,\beta) + \varepsilon^2 x_2 (\alpha,\beta) + \cdots \\
t &= t_o (\alpha,\beta) + \varepsilon t_1 (\alpha,\beta) + \varepsilon^2 t_2 (\alpha,\beta) + \cdots
\end{aligned}
$$

These series are substituted in the system of Equations (2) and the equations are separated according to powers in ε as is the standard procedure. Since the flow field is uniform to lowest order, the equations up to second order become:

u_o = constant; a_o = constant = 1

$$\frac{\partial \psi_0}{\partial \alpha} = (u_0+1)\frac{\partial t_0}{\partial \alpha}\ ;\ \frac{\partial \psi_0}{\partial \beta} = (u_0-1)\frac{\partial t_0}{\partial \beta} \qquad (3)$$

$$\frac{2}{\gamma-1}\ \frac{\partial a_1}{\partial \alpha} + \frac{\partial u_1}{\partial \alpha} = 0\ ;\ \frac{2}{\gamma-1}\ \frac{\partial a_1}{\partial \beta} - \frac{\partial u_1}{\partial \beta} = 0 \qquad (4)$$

$$\begin{aligned}\frac{\partial \psi_1}{\partial \alpha} &= (u_0+1)\ \frac{\partial t_1}{\partial \alpha} + (u_1+a_1)\ \frac{\partial t_0}{\partial \alpha}\\ \frac{\partial \psi_1}{\partial \beta} &= (u_0-1)\ \frac{\partial t_1}{\partial \beta} + (u_1-a_1)\ \frac{\partial t_0}{\partial \beta}\end{aligned} \qquad (5)$$

$$\frac{2}{\gamma-1}\ \frac{\partial a_2}{\partial \alpha} + \frac{\partial u_2}{\partial \alpha} = 0\ ;\ \frac{2}{\gamma-1}\ \frac{\partial a_2}{\partial \beta} - \frac{\partial u_2}{\partial \beta} = 0 \qquad (6)$$

First, the Equations (4) and (6) for u and a are solved. The first and second order equations are similar so they will be solved in identical fashion. Letting the subscript i = 1 or 2, the solutions are:

$$\frac{2}{\gamma-1}\ a_i + u_i = P_i(\beta)$$
$$\frac{2}{\gamma-1}\ a_i - u_i = Q_i(\alpha)$$

so that

$$\begin{aligned} u_i &= \frac{P_i(\beta) - Q_i(\alpha)}{2}\\ a_i &= \frac{\gamma-1}{4}\left[P_i(\beta) + Q_i(\alpha)\right]\end{aligned} \qquad (7)$$

Note that P/2 and Q/2 are the Riemann invariants. (Reference 17).

The P_1 (β) and Q_1 (α) are still arbitrary and may be determined by a knowledge of initial conditions and boundary conditions. The boundary conditions are well-defined, as will later be shown, but there is no knowledge of initial conditions. The solution will be obtained by leaving the initial conditions arbitrary and applying the cyclic condition in its place. Once P_1 and Q_1 are determined, a simple calculation yields a_1 and u_1. Also, once these are determined, x and t are found from their governing differential equations and boundary conditions which allows the transformation to the original coordinate system.

Since the nozzle is short any oscillatory processes within it may be considered quasi-steady. Therefore, the entrance Mach number is constant since it is a function of area ratio only. The nozzle boundary condition becomes

$$u_i = u_o \; a_i \quad at \quad \alpha' = 1 + \beta' \tag{8}$$

or $$Q_i' = \frac{1-\nu}{1+\nu} \; P_i' \quad at \quad \alpha' = 1+\beta'$$

where $$\nu \equiv \frac{\gamma-1}{2} \; u_o$$

Equation (1) in nondimensional form gives the boundary condition at the combustion zone.

$$u = u_o + \omega u_o \; (a-1) + \delta u_o \; (a-1)^2 + O(a-1)^3 \tag{9}$$

This results in the following boundary condition for the first order terms in the perturbation series

$$u_1 = \omega u_o \; a_1$$

When $\omega > 1$, the admittance at the concentrated combustion zone is larger than the nozzle admittance. This is an unstable situation as can be shown by a small perturbation analysis which indicates an exponential growth of wave amplitude when $\omega > 1$. This small perturbation analysis is standard and is performed in Appendix C. It follows from a.

comparison with the small perturbation analysis that an oscillation cannot be periodic to first order in ε if no shock is present since the energy added to the oscillation by the interaction with the combustion process is greater than the energy taken from the oscillation by the nozzle outflow. If a periodic solution is to exist there must be another mechanism besides the nozzle which removes energy of oscillation. Since a shock wave seems to be the most realistic choice of such a dissipative mechanism, its existence has been postulated. The dissipation of energy by a shock is monotonically increasing with its amplitude or strength. More specifically, the dissipation or creation of entropy is of third order in shock strength. It is then expected that the strength of the shock is monotonically increasing with the difference in the admittances of the combustion zone and the nozzle. Note, of course, that ε is representative of the shock strength and $(\omega-1)$ is representative of the difference in the admittances at the chamber ends. Since series convergence cannot be proven for any choice of ε, there is freedom in the specification of the relationship between ε and $(\omega-1)$. Therefore, the most simple functional relationship will be chosen which says that ε is directly proportional to $(\omega-1)$ or specifically

$$\varepsilon = \nu(\omega - 1) \tag{10}$$

The boundary condition (9) at the combustion zone now becomes

$$u_i = u_o \; a_i + R_i \quad \text{at} \quad x = 0$$

where

$$R_1 \equiv 0$$

$$R_2 \equiv \frac{2}{\gamma-1} \, a_1 + u_o \, \delta \, a_1^2$$

This may be put in the more convenient form

$$P_i = \frac{1+\nu}{1-\nu} Q_i + \frac{\ell}{1-\nu} R_i \tag{11}$$

To first order the boundary conditions at the chamber ends are identical so that a periodic solution may now be obtained.

As is well known one family of the Riemann invariants is always continuous up to second order through a shock wave; P invariants may be continued through rearward-moving shocks while Q invariants may be continued through forward-moving shocks. The two boundary conditions (8) and (11) and this property of continuation through shocks allows the determination of P_1 (β), P_1' (β'), and Q_1' (α') in terms of Q_1 (α). It is found that

$$\begin{aligned} P_i(\beta) &= \frac{1+\nu}{1-\nu} Q_i(\beta) + \frac{\ell}{1-\nu} R_i(\beta) \\ P_i'(\beta') &= P_i(\beta) \quad \text{where} \quad \beta=\beta'; \text{ i.e., } P_i'(\beta) = P_i(\beta) \\ Q_i'(\alpha') &= \frac{1-\nu}{1+\nu} P_i'(\alpha'-1) = Q_i(\alpha'-1) + \frac{\ell}{1+\nu} R_i(\alpha'-1) \end{aligned} \tag{12}$$

Note that Q_1 (α) will be determined upon application of the cyclic condition.

When the cyclic condition is applied to determine Q_1, it is necessary to relate α and β to x and t. Hence, the Equations (3) and (5) for x and t must be solved. The zero order Equations (3) may be put in the convenient form.

$$\frac{\partial}{\partial\beta}\left[\frac{x_0}{1-u_0} + t_0\right] = 0; \quad \frac{\partial}{\partial\alpha}\left[\frac{x_0}{1+u_0} - t_0\right] = 0$$

with the general solution

$$\frac{x_0}{1-u_0} + t_0 = G_0(\alpha); \qquad \frac{x_0}{1+u_0} - t_0 = F_0(\beta)$$

The boundary conditions state

$$x_0 = 0 \text{ at } \alpha = \beta$$
$$x_0 = 1 \text{ at } \alpha' = 1 + \beta'$$
$$t_0 = 0 \text{ at } \alpha = \beta = 0$$
$$x_0 = \alpha \text{ at } \beta = 0$$

Upon their application, it is found that for Region I

$$G_0(\alpha) = \frac{2\alpha}{1-u_0^2} \quad ; \quad F_0(\beta) = -\frac{2\beta}{1-u_0^2} \tag{13a}$$
$$x_0 = \alpha - \beta \quad ; \quad t_0 = \frac{\alpha}{1+u_0} + \frac{\beta}{1-u_0}$$

Using primes on x, t, F, G, α, and β in the above would give the solution for Region II

$$G_0'(\alpha') = \frac{2\alpha'}{1-u_0^2} \quad ; \quad F_0'(\beta') = -\frac{2\beta'}{1-u_0^2} \tag{13b}$$
$$x_0' = \alpha' - \beta' \quad ; \quad t_0' = \frac{\alpha'}{1+u_0} + \frac{\beta'}{1-u_0}$$

Similarly, the first order Equations (5) may be written as

$$\frac{\partial}{\partial\beta}\left[\frac{x_1}{1-u_0} + t_1\right] = \frac{3-\gamma}{4}\,\frac{P_1(\beta)}{(1-u_0)^2} - \frac{\gamma+1}{4}\,\frac{Q_1(\alpha)}{(1-u_0)^2}$$

$$\frac{\partial}{\partial\alpha}\left[\frac{x_1}{1+u_0} - t_1\right] = \frac{\gamma+1}{4}\,\frac{P_1(\beta)}{(1+u_0)^2} - \frac{3-\gamma}{4}\,\frac{Q_1(\alpha)}{(1+u_0)^2}$$

with the general solution for Region I

$$\frac{x_1}{1-u_0} + t_1 = G_1(\alpha) + \frac{3-\gamma}{4}\left(\frac{1}{1-u_0}\right)^2 \int_0^\beta P_1(z)\,dz - \frac{\gamma+1}{4}\left(\frac{1}{1-u_0}\right)^2 \beta\, Q_1(\alpha)$$

$$\frac{x_1}{1+u_0} - t_1 = F_1(\beta) + \frac{\gamma+1}{4}\left(\frac{1}{1+u_0}\right)^2 \alpha\, P_1(\beta) - \frac{3-\gamma}{4}\left(\frac{1}{1+u_0}\right)^2 \int_0^\alpha Q_1(z)\,dz \tag{14a}$$

where z is a dummy variable. For Region II the solution is

$$\frac{x_1'}{1-u_0} + t_1' = G_1'(\alpha') + \frac{3-\gamma}{4}\left(\frac{1}{1-u_0}\right)^2 \int_0^{\beta'} P_1'(z)\,dz - \frac{\gamma+1}{4}\left(\frac{1}{1-u_0}\right)^2 \beta' Q_1'(\alpha')$$

$$\frac{x_1'}{1+u_0} - t_1' = F_1'(\beta') + \frac{\gamma+1}{4}\left(\frac{1}{1+u_0}\right)^2 \alpha' P_1'(\beta') - \frac{3-\gamma}{4}\left(\frac{1}{1+u_0}\right)^2 \int_1^{\alpha'} Q_1'(z)\,dz \qquad (14b)$$

The initial condition would be

$$\frac{x_1}{1-u_0} + t_1 = G_1(\alpha) \quad \text{at } \beta = 0$$

However, the initial condition will be left arbitrary and will be replaced by a cyclic condition. That is, G_1 is left arbitrary. The boundary conditions are

$$t_1 = 0 \text{ at } \alpha = \beta = 0$$
$$x_1 = 0 \text{ at } \alpha = \beta$$
$$x_1' = 0 \text{ at } \alpha' = 1 + \beta' \qquad (15a)$$

The convenience of grouping the variables x_1 and t_1 into the two families shown above will be demonstrated when the solutions are continued across shock waves. These two families possess a similar property to the Riemann invariants; that is, one of the two families may always be continued across a shock discontinuity. As is shown in Figure 4a, characteristic lines are constantly intersecting shock waves from both sides. Of course, they no longer exist in a real fashion after their intersection. When using characteristic coordinates, it is convenient to extend the characteristics beyond their points of intersection with the shock by considering the transformation of x, t to α, β as being double-valued for some small region ($O[\varepsilon]$) near the shock wave. Of course, only one value of the transformation is applicable in reality, but this abstraction saves much labor. Figure 4b shows the slight overlapping of the α, β sheet (Region I) and the α', β' sheet (Region II) near the rearward-moving shock BC. Now, the Riemann invariant $P(\beta)/2$ may be continued across the shock by saying $P_1(\beta)$ at point O (see Figure 4c) equals $P_1'(\beta')$ at point O' where $\beta = \beta'$ by definition since they meet at the shock wave. It is immediately shown by a Taylor series expansion about O and O' and a matching process at the shock that

$$\left[\frac{x_1}{1+u_0} - t_1\right]_O = \left[\frac{x_1'}{1+u_0} - t_1'\right]_{O'} + O(\varepsilon^2) \tag{15b}$$

As shown schematically in Figure 4b, the path of the shock BC is given by the relationship:

$$\alpha = 1 + \varepsilon\eta(\beta) + O(\varepsilon^2)$$

This applies in Region I. The equivalent statement written for Region II says:

$$\alpha' = 1 + \varepsilon\xi(\beta') + O(\varepsilon^2)$$

Obviously, the function $\left[\frac{x}{1+u_0} - t\right]$ is

continuous across the shock wave since x and t are continuous across the shock wave. It follows that this function has the same value at the shock in both Region I and Region II. This function may be expressed as a Taylor series in each region and then the two series are set equal at the shock wave. In Region I, the function is expanded about the line $\alpha = 1$ (point O) and, in Region II, about the line $\alpha' = 1$ (point O'). The final result is:

$$\left[\frac{x_0}{1+u_0} - t_0\right]_O + \varepsilon\left[\frac{\partial}{\partial\alpha}\left(\frac{x_0}{1+u_0} - t_0\right)\right]_O \eta(\beta) + \varepsilon\left[\frac{x_1}{1+u_0} - t_1\right]_O$$

$$= \left[\frac{x_0'}{1+u_0} - t_0'\right]_{O'} + \varepsilon\left[\frac{\partial}{\partial\alpha}\left(\frac{x_0'}{1+u_0} - t_{0'}\right)\right]_{O'} \xi(\beta') + \varepsilon\left[\frac{x_1'}{1+u_0} - t_1'\right]_{O'} + O(\varepsilon^2)$$

Note that $\frac{\partial}{\partial\alpha}\left[\frac{\psi_0}{1+u_0}-t_0\right]$ is zero everywhere and that, as can be shown from Equation (13), $\left[\frac{\psi_0}{1+u_0}-t_0\right]_0=\left[\frac{\psi_0'}{1+u_0}-t_0'\right]_{0'}$.

From this, it is seen that Equation (15b) immediately follows. Then Equations (14) and (15b) yield $F_1(\beta)=F_1'(\beta')+\frac{3-\gamma}{4}\left(\frac{1}{1+u_0}\right)^2\int_0^1 Q_1(z)dz+O(\epsilon^2)$

The advantage of this is that it is not necessary to calculate explicitly the exact location of our shock before proceeding with the determination of higher order coefficients in Region II. This method has uncoupled that calculation from the rest, and it may be performed later, if so desired, providing a tremendous simplification over previous methods. Similarly, the other family or grouping of x_1 and t_1 may be continued across a forward-moving shock wave. This continuation process is not valid for x_2 and t_2 where discontinuities will appear.

The conditions (15a) and (15b) lead to the following:

$$\begin{aligned}
F_1(\beta) &= -G_1(\beta)+\frac{3-\gamma}{4}\left\{\left(\frac{1}{1+u_0}\right)^2-\frac{1+v}{1-v}\left(\frac{1}{1-u_0}\right)^2\right\}\int_0^\beta Q_1(z)dz \\
&\quad+\frac{\gamma+1}{4}\left\{\left(\frac{1}{1-u_0}\right)^2-\frac{1+v}{1-v}\left(\frac{1}{1+u_0}\right)^2\right\}Q_1(\beta)\beta \\
F_1'(\beta) &= F_1(\beta)-\frac{3-\gamma}{4}\left(\frac{1}{1+u_0}\right)^2\int_0^1 Q_1(z)dz+O(\epsilon^2) \qquad (16)\\
G_1'(\alpha') &= G_1(\alpha'-1)+\frac{3-\gamma}{4}\left(\frac{1}{1+u_0}\right)^2\int_0^1 Q_1(z)dz \\
&\quad-\frac{\gamma+1}{4}\left(\frac{1}{1+u_0}\right)^2\frac{1+v}{1-v}Q_1(\alpha'-1)+O(\epsilon^2)
\end{aligned}$$

The cyclic condition is applied only by stating that flow conditions along the shock AB in Region I are identical to conditions along the shock CD in Region III. When the cyclic condition is applied, it is necessary to know the difference in the values of α' and α'' at any point of their intersection with the shock wave CD.

In Region III, the same differential Equations (3) and (5) govern x and t with the boundary conditions:

$$\alpha'' = x \quad \text{at} \quad \beta'' = 0$$
$$x = 0 \quad \text{at} \quad \alpha'' = \beta''$$
$$t = T \quad \text{at} \quad \alpha'' = \beta'' = 0$$

where T is the period of the oscillation and is still unknown. These must result in the following for the cyclic condition to hold.

$$G_0''(\alpha'') = G_0(\alpha'') + T_0$$
$$G_1''(\alpha'') = G_1(\alpha'') + T_1 \tag{17}$$

where $T = T_0 + \varepsilon T_1 + \text{-----}$

Before proceeding the period must be determined in order to obtain the solution for Region III. It is known that the period is directly related to shock velocities by the relations

$$\int_0^1 d\psi = \int_0^{T'} V_{AB}\, dt \; ; \; \int_1^0 d\psi = \int_{T'}^{T'+T''} V_{BC}\, dt \tag{18}$$

where $T = T' + T''$ and T' is the time of forward-shock travel and T'' is the time of rearward-shock travel. It is also known that the shock velocities may be related to the flow properties on both sides of the shock by means of the conservation equations. A well-known result is that the instantaneous velocity for a forward-travelling shock is approximately (to first order) the average of the slopes of the two P-characteristics (one on each side in an x vs. t plot) intersecting the shock at that instant. For rearward-moving shocks, a similar approximation uses the slopes of Q-characteristics. The slopes $\frac{d\phi}{dt} = u \pm a$ may be related* to Q through Equations (7) and (12). In a convenient form these approximations are:

$$V_{AB} = 1 + u_0 + \varepsilon \left\{ -\frac{3-\gamma}{4} Q_1(\alpha) + \frac{\gamma+1}{8} \frac{1+\nu}{1-\nu} \left[Q_1(1) + Q_1(0) \right] \right\} + o(\varepsilon^2)$$

$$V_{BC} = -1 + u_0 + \varepsilon \left\{ \frac{3-\gamma}{4} \frac{1+\nu}{1-\nu} Q_1(\beta) - \frac{\gamma+1}{8} \left[Q_1(1) + Q_1(0) \right] \right\} + o(\varepsilon^2) \tag{19}$$

Combination of Equations (18), and (19) and separation according to powers of ε yield the following results:

$$1 = (1+u_0)\, T_0'$$

$$-1 = (u_0 - 1)\, T_0''$$

$$0 = (1+u_0)\, T_1' + \int_0^1 \left\{ -\frac{3-\gamma}{4} Q_1(\alpha) + \frac{\gamma+1}{8} \frac{1+\nu}{1-\nu} \left[Q_1(1) + Q_1(0) \right] \right\} \frac{d\alpha}{1+u_0}$$

$$0 = (u_0 - 1)\, T_1'' + \int_0^1 \left\{ \frac{3-\gamma}{4} Q_1(\beta) - \frac{\gamma+1}{8} \left[Q_1(1) + Q_1(0) \right] \right\} \frac{d\beta}{1-u_0}$$

These four relationships determine T_0', T_0'', T_1', and T_1'' which, upon addition, yield the results for T_0 and T_1.

$$T_0 = \frac{2}{1-u_0^2}$$

$$T_1 = \frac{3-\gamma}{4} \left[\left(\frac{1}{1+u_0}\right)^2 + \frac{1+\nu}{1-\nu} \left(\frac{1}{1-u_0}\right)^2 \right] \int_0^1 Q_1(\beta)\, d\beta - \frac{\gamma+1}{8} \left[\left(\frac{1}{1+u_0}\right)^2 \frac{1+\nu}{1-\nu} + \left(\frac{1}{1-u_0}\right)^2 \right] \left[Q_1(1) + Q_1(0) \right]$$

These relationships for T_0 and T_1 are substituted into Equation (17) with the result:

$$G_0''(\alpha'') = G_0(\alpha'') + \frac{2}{1-u_0^2}$$

$$G_1''(\alpha'') = G_1(\alpha'') + \frac{3-\gamma}{4} \left[\left(\frac{1}{1+u_0}\right)^2 + \frac{1+\nu}{1-\nu} \left(\frac{1}{1-u_0}\right)^2 \right] \int_0^1 Q_1(\beta)\, d\beta - \frac{\gamma+1}{8} \left[\left(\frac{1}{1+u_0}\right)^2 \frac{1+\nu}{1-\nu} + \left(\frac{1}{1-u_0}\right)^2 \right] \left[Q_1(1) + Q_1(0) \right] \tag{20}$$

*Note that in order to determine the velocity of shock AB, flow conditions immediately in front of shock AB are set equal to flow conditions immediately in front of shock CD. This is valid because of the cyclic nature of the phenomenon.

Now, the function $\frac{\psi}{1-u_0}+t$ is continuous (up to and including first order) through the shock CD. It is evaluated in Region II from Equations (13b), (14b), and (16) and in Region III from Equations (13a), (14a), and (20). Note that the function $\frac{\psi''}{1-u_0}+t''$ in Region III is identical to $\frac{\psi}{1-u_0}+t$ in Region I plus a constant equal to the period of oscillation. The result of matching this function across the shock CD is

$$\frac{2(1+\alpha'')}{1-u_0^2}+\varepsilon\left\{G_1(\alpha'')+\frac{3-\gamma}{4}\left[\left(\frac{1}{1+u_0}\right)^2+\frac{1+v}{1-v}\left(\frac{1}{1-u_0}\right)^2\right]\int_0^1 Q_1(z)\,dz\right.$$
$$\left.-\frac{\gamma+1}{8}\left[\left(\frac{1}{1+u_0}\right)^2\frac{1+v}{1-v}+\left(\frac{1}{1-u_0}\right)^2\right]\left[Q_1(1)+Q_1(0)\right]\right\}=\frac{2\alpha'}{1-u_0^2}$$
$$+\varepsilon\left\{G_1(\alpha'-1)+\frac{3-\gamma}{4}\left[\left(\frac{1}{1+u_0}\right)^2+\frac{1+v}{1-v}\left(\frac{1}{1-u_0}\right)^2\right]\int_0^1 Q_1(z)\,dz\right.$$
$$\left.-\frac{\gamma+1}{4}\left[\left(\frac{1}{1+u_0}\right)^2\frac{1+v}{1-v}+\left(\frac{1}{1-u_0}\right)^2\right]Q_1(\alpha'-1)\right\}+O(\varepsilon^2)$$

It is readily seen that $\alpha''-\alpha'+1$ is of order ε. Therefore, $\varepsilon\left[G_1(\alpha'')-G_1(\alpha'-1)\right]$ and $\varepsilon\left[Q_1(\alpha'')-Q_1(\alpha'-1)\right]$ are both of order ε^2. Using this, a simple relationship results for the difference in the values of α'' and α' at the shock CD. The knowledge of this difference is necessary for the application of the cyclic condition as will later be shown. This difference is the "slippage" of the characteristics at the shock CD and is found to be the following:

$$\varepsilon\Delta\alpha\equiv\alpha''-\alpha'+1=$$
$$=\varepsilon\,\frac{\gamma+1}{8}\left[\frac{1-u_0}{1+u_0}\,\frac{1+v}{1-v}+\frac{1+u_0}{1-u_0}\right]\left\{\frac{Q_1(1)+Q_1(0)}{2}-Q_1(\alpha'')\right\}+O(\varepsilon^2)$$

This "slippage" $\varepsilon\,\Delta\alpha$ is shown schematically in Figure 5.

The continuation of the Riemann invariant across the shock CD results in:

$$\left[\varepsilon Q_1''(\alpha'')+\varepsilon^2 Q_2''(\alpha'')\right]_{III}=\left[\varepsilon Q_1'(\alpha')+\varepsilon^2 Q_2'(\alpha')\right]_{II}+O(\varepsilon^3)$$

and noting that $\alpha'' = \alpha' - 1 + \varepsilon\,\Delta\alpha$ we get

$$\left[\varepsilon Q_1''(\alpha'-1) + \varepsilon^2 \frac{dQ_1''}{d\alpha'}(\alpha'-1)\Delta\alpha + \varepsilon^2 Q_2''(\alpha'-1)\right]_{III} =$$

$$\left[\varepsilon Q_1'(\alpha') + \varepsilon^2 Q_2'(\alpha')\right]_{II} + O(\varepsilon^3)$$

The cyclic condition says that the flow in Region I is identical to that in Region III so that

$$\varepsilon Q_1(\alpha'-1) + \varepsilon^2 \frac{dQ_1}{d\alpha}(\alpha'-1)\Delta\alpha + \varepsilon^2 Q_2(\alpha'-1) =$$

$$\varepsilon Q_1'(\alpha') + \varepsilon^2 Q_2'(\alpha') + O(\varepsilon^3)$$

Noting the relation between $Q_i'(\alpha')$ and $Q_i(\alpha)$ as given by Equation (12), it is found that our matching condition is trivial to first order and gives a governing ordinary differential equation for $Q_1(\alpha)$ to second order

$$\frac{dQ_1}{d\alpha}\,\Delta\alpha = \frac{2}{1+\nu}\,R_2(\alpha) \tag{21}$$

The solution to this equation will yield the solution of our problem. What has been done is very common in nonlinear near-resonant solutions; that is, a second order investigation has yielded a first order solution. If rotational effects did not make it impossible, a third order investigation would yield the second order solution and so forth. However, a first order solution yields a good deal of qualitative information about the oscillation and also, for not-too-large amplitudes, the analysis is quite accurate in a quantitative manner assuming there exist no other errors than those related to the degree of the approximation.

If the following definitions are made

$$\mu = \frac{16}{\gamma+1}\;\frac{1}{(1-\nu^2)\left[\frac{1-u_0}{1+u_0}\,\frac{1+\nu}{1-\nu} + \frac{1+u_0}{1-u_0}\right]}$$

$$\lambda = \frac{16 u_0 \delta}{(\gamma+1)(1+\nu)}\left[\left(\frac{\gamma-1}{2}\right)\frac{1}{1-\nu}\right]^2 \Big/ \left[\frac{1-u_0}{1+u_0}\,\frac{1+\nu}{1-\nu} + \frac{1+u_0}{1-u_0}\right]$$

$$C = \frac{Q_1(1) + Q_1(0)}{2} \tag{22}$$

then Equation (21) becomes

$$\frac{dQ_1}{d\alpha} = - \frac{rQ_1 + \lambda Q_1^2}{Q_1 - C}$$

This first order nonlinear ordinary differential equation normally requires one boundary condition for the complete solution, but there is none. Also, the values of the function at the endpoints of the domain of interest $0 \leq \alpha \leq 1$ appear as constants in the equation. These constants are not known apriori. These two difficulties will be overcome in the following manner. It can be shown that physically reasonable solutions exist only for one value of the parameter C. That only C = 0 can give an oscillatory solution is the important result of a topological investigation of the ordinary differential equation (Appendix D). Also, stating C=0 gives a relation between the values of the solution at two points α =0 and α =1. This is just as satisfactory as stating the definite value of the solution at one point which is the usual statement of the initial or boundary condition. Hence, the following solution is obtained for Equation (21).

$$Q_1(\alpha) = \frac{r}{\lambda}\left[\frac{2}{1+e^{-\lambda}} \quad e^{-\lambda\alpha} - 1\right] \tag{23}$$

Note the exponential form of the solution indicates an exponential wave form. Also, as $\lambda \rightarrow 0 \; (\delta \rightarrow 0)$ a sawtooth wave form is obtained as shown by the limiting case

$$Q_1(\alpha) = r\left(\frac{1}{2} - \alpha\right)$$

Note that once $Q_1(\alpha)$ is determined $P_1(\beta)$, $Q_1'(\alpha')$, and $P_1'(\beta')$ may be determined from Equation (12) so that sufficient information is available to calculate the flow field properties from Equations (7, 13) as a function of position and time. Note that the pressure perturbation may easily be related to the speed of sound perturbation by the isentropic relation. For Region I, the final solutions are obtained

$$u-u_0 = \varepsilon \frac{r}{\lambda}\left\{\frac{e^{-\lambda\left(\frac{1-u_0^2}{2}\right)t}}{1+e^{-\lambda}}\left[\frac{1+\nu}{1-\nu}\, e^{\frac{\lambda}{2}(1-u_0)x} - e^{-\frac{\lambda}{2}(1+u_0)x}\right] - \frac{\nu}{1-\nu}\right\} + O(\varepsilon^2)$$

$$P-P_0 = \varepsilon\gamma \frac{r}{\lambda}\left\{\frac{e^{-\lambda\left(\frac{1-u_0^2}{2}\right)t}}{1+e^{-\lambda}}\left[\frac{1+\nu}{1-\nu}\, e^{\frac{\lambda}{2}(1-u_0)x} + e^{-\frac{\lambda}{2}(1+u_0)x}\right] - \frac{1}{1-\nu}\right\} + O(\varepsilon^2)$$

and for Region II, the solutions are

$$u-u_0 = \varepsilon \frac{r}{\lambda}\left\{\frac{e^{-\lambda\left(\frac{1-u_0^2}{2}\right)t}}{1+e^{-\lambda}}\left[\frac{1+\nu}{1-\nu}\, e^{\frac{\lambda}{2}(1-u_0)x} - e^{-\frac{\lambda}{2}(1+u_0)x}\, e^{\lambda}\right] - \frac{\nu}{1-\nu}\right\} + O(\varepsilon^2)$$

$$P-P_0 = \varepsilon\gamma \frac{r}{\lambda}\left\{\frac{e^{-\lambda\left(\frac{1-u_0^2}{2}\right)t}}{1+e^{-\lambda}}\left[\frac{1+\nu}{1-\nu}\, e^{\frac{\lambda}{2}(1-u_0)x} + e^{-\frac{\lambda}{2}(1+u_0)x}\, e^{\lambda}\right] - \frac{1}{1-\nu}\right\} + O(\varepsilon^2)$$

The shock strength is calculated from the above relations to be as follows:

$$\Delta u_{AB} = \varepsilon \frac{r}{\lambda}\,\frac{1+\nu}{1-\nu}\,\frac{1-e^{-\lambda}}{1+e^{-\lambda}} + O(\varepsilon^2)$$

$$\Delta u_{BC} = \varepsilon \frac{r}{\lambda}\,\frac{1-e^{-\lambda}}{1+e^{-\lambda}} + O(\varepsilon^2)$$

$$\Delta P_{AB} = \varepsilon\gamma \frac{r}{\lambda}\,\frac{1+\nu}{1-\nu}\,\frac{1-e^{-\lambda}}{1+e^{-\lambda}} + O(\varepsilon^2)$$

$$\Delta P_{BC} = \varepsilon\gamma \frac{r}{\lambda}\,\frac{1-e^{-\lambda}}{1+e^{-\lambda}} + O(\varepsilon^2)$$

Also, the shock velocities are

$$V_{AB} = 1+u_0 + \varepsilon \frac{r}{\lambda}\left\{\frac{3-\gamma}{4} - \frac{3-\gamma}{2}\,\frac{e^{-\lambda(1+u_0)t}}{1+e^{-\lambda}}\right\} + O(\varepsilon^2)$$

$$V_{BC} = u_0 - 1 + \varepsilon \frac{r}{\lambda}\,\frac{1+\nu}{1-\nu}\left\{\frac{\gamma-3}{4} + \frac{3-\gamma}{2}\,\underline{e^{\lambda\left(\frac{1}{1+u_0}-t\right)(1-u_0)}}\right\} + O(\varepsilon^2)$$

and the period of oscillation is:

$$T = \frac{2}{1-u_0^2} + \varepsilon\,\frac{3-\gamma}{4}\,\frac{r}{\lambda}\left[\frac{1+\nu}{1-\nu}\left(\frac{1}{1-u_0}\right)^2 + \left(\frac{1}{1+u_0}\right)^2\right]\left\{\frac{2}{\lambda}\,\frac{1-e^{-\lambda}}{1+e^{-\lambda}} - 1\right\} + O(\varepsilon^2)$$

The solution indicates a shock discontinuity followed by an exponential decay as shown in Figure 6. For small λ this decay is nearly linear (sawtooth type). There is no difficulty as $\lambda \rightarrow 0$ since $\frac{1-e^{-\lambda}}{\lambda}$ stays finite. The shock strengths stay constant with time and strength is lost or gained only in reflection. The absolute value of the shock velocity always increases in time and the perturbation on the natural period of oscillation can be shown to be of the order of $\varepsilon\left(\frac{\lambda}{12}\right)$ which is usually negligible. As $\varepsilon \rightarrow 0$ (or $\omega \rightarrow 1$) the shock strength goes to zero. This is an important result since it shows a consistency between linear and nonlinear results; that is, the same stability limit is predicted by both linear and nonlinear analyses. The main effect on amplitude is produced through ε while λ has only a secondary effect on amplitude. The importance of λ is that it indicates a definite relationship between the forcing function of the instability and the wave form of the oscillation.

NUMERICAL EXAMPLE

A special case of the energy release rate r in Appendix B will be examined. In this case

$$r = e^{-E/RT^*}$$

where E/R is a parameter and T* is considered as dimensional. This rate function admittedly may be too simple to be realistic, but is primarily intended to show the interesting relationship between functional forms of the combustion laws and the flow oscillation in a quantitative manner.

ϵ and λ may be calculated by use of Equations (10), (22), and (B-9). The following are obtained

$$\varepsilon = u_0 \left[(\gamma - 1) \frac{E}{RT^*} - \frac{3\gamma - 1}{2} \right]$$

$$\lambda = \frac{16 u_0 \left(\frac{\gamma - 1}{\gamma} \frac{1}{1 - \nu} \right)^2 \left[2 \left(\frac{E}{RT^*} \right)^2 - \frac{7\gamma - 3}{\gamma - 1} \frac{E}{RT^*} + \frac{\gamma (3\gamma - 1)}{(\gamma - 1)^2} \right]}{(\gamma + 1)(1 + \nu) \left[\frac{1 - u_0}{1 + u_0} \frac{1 + \nu}{1 - \nu} + \frac{1 + u_0}{1 - u_0} \right]}$$

Using the values $\gamma = 1.2$ and $u_o = .1$ the following results:

$\frac{E}{RT^*}$	ε	λ	r	ΔP_{shock}
10.0	.07	.028	3.53	.152
12.5	.12	.189	3.53	.259
15.0	.17	.439	3.53	.362
17.5	.22	.778	3.53	.453
20.0	.27	1.207	3.53	.522

The table shows that for this case, the combustor is most unstable for low temperature operation and most stable in high temperature operation. The amplitudes are not excessively large and the exponential shape of the wave is not too severe despite the exponential form of the combustion law. Figure 7 indicates the pressure wave shape at both ends of the chamber over the period of oscillation for the values of $\lambda = .6$ and 1.2. Note that the shape is not very different from a sawtooth wave.

Although calculations have been made here only for a very simple case, there is no reason why they cannot be accomplished for more realistic combustion processes.

CHAPTER III

NONLINEAR COMBUSTION INSTABILITY WITH TIME-LAG EFFECTS: LONGITUDINAL MODES

FORMULATION OF THE PROBLEM

A model of the longitudinal modes of instability is investigated in which the characteristic time of the combustion process is of the same order as the wave travel time in the combustion chamber. This is different from the model of Chapter II where the time of combustion was negligible compared to the wave travel time. When the combustion time and the wave travel time are of the same order, the phasing between pressure (or gas velocity) oscillation and the energy (or mass) addition oscillation is affected. One important result of this phasing effect is that the frequency of the oscillation may be different from a natural resonant frequency of the chamber (See Ref. 1). (In the model of Chapter II, the resonant frequency was found). On account of this result, there are important qualitative differences between this case and the case where the combustion process is instantaneous (Chapter II). Here, as will later be shown, the possibility of broadening the range of unstable operating conditions by nonlinear effects exists. This is related to the "triggering" action explained in Chapter I. Another possibility is that stable periodic solutions with finite amplitudes and without shock waves are possible. This was not possible in the model of Chapter II where the additional dissipative mechanism was necessary to maintain periodicity. The approach of the analysis of Chapter III will be to find a periodic solutions of finite amplitude without shock waves and, then, determine the stability of this periodic solution. Instability of the periodic solution will indicate the possibility of "triggering" action.

Many of the assumptions made are identical to some of those of Chapter II and produce similar simplications in the analysis. The assumptions are:

(1) The flow is one-dimensional.

(2) The chamber cross-sectional area is constant.

(3) The chamber is very long so that the configuration is well-approximated by the limiting cases of zero-length convergent portion of nozzle and of concentrated combustion zone at the injector end of the chamber.

(4) Flow is homentropic; i.e., there are no shock waves and no entropy waves in the chamber.

(5) The gas in the chamber is calorically perfect.

(6) The Crocco time-lag postulate is made. Through this postulate, the characteristic time of the combustion process is introduced to the analysis.

The time-lag postulate is presented in Ref. 1 (where a linearized theory is developed) and is extended to include nonlinear terms in Appendix E. According to Crocco, the concept is that the rate f of the combustion process (as experienced by any particle) is sensitive to the fluctuations of thermodynamic properties over a period* of time τ prior to the instant t at which the particle becomes burned chamber gas. It is assumed that the time integral of the rate function over the period τ is constant. In other words, some entity is accumulated at the rate f until at time t, the critical amount E necessary for complete combustion has been accumulated. τ is the period of accumulation such that

$$E = \int_{t-\tau}^{t} f \, dt'$$

The nature of the entity is never specified and the relationship between f and the thermodynamic properties is not known exactly. So, clearly, this is a heuristic approach.

The fluctuation in f produces a fluctuation in τ which allows the value of the integral to be constant. In particular, the more rapid the combustion process (f increases), the shorter the time of the combustion process (τ decreases) and vice versa. Of course, the fluctuation in f is related to the fluctuations in pressure and temperature. The fluctuation of τ, therefore, is related to the fluctuations of the thermodynamic properties at the combustion zone.

The fluctuation of the mass flux emitted from the combustion zone at any instant t is related to the period τ associated with the particle completing combustion and being emitted from the combustion zone at time t. The specific re-

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* Note that τ is actually a time period or duration of the pressure (and temperature) sensitive portion of the combustion process. It will appear in the equations later as a time-lag only due to the perturbation scheme.

lationship as developed in Ref. 1 and Appendix E is

$$\frac{\dot{m}_b(t)}{\dot{m}_i} = 1 - \frac{d\tau(t)}{dt}$$

where $\dot{m}_b$ is the mass flux of burned gases emitted from the combustion zone and $\dot{m}_i$ is the mass flux injected into the chamber (assumed constant).

The fluctuation in mass flux $\dot{m}_b$ may now be related to the fluctuation of pressure and temperature. This relationship describes the feedback of energy to the oscillation. Details of the formulation are presented in Appendix E. The result expressed by Equation (E-17) will be used as the boundary condition at the injector end of the chamber. This boundary condition is needed to solve the governing equations of the instability phenomenon in the combustion chamber.

Naturally this heuristic approach leaves much to be desired. It can only be justified on the following bases:

1) At this time, the nature of the combustion process is not completely understood even for steady-state operation. In particular, the manner in which energy is fed-back into the oscillation of the chamber gases by the combustion process is not known for certain.

2) The linearized theories of Crocco, Cheng, and Reardon based on this same approach have been quite successful in predicting the magnitudes of the off-resonant frequencies and the location of the stability limits. Agreement between theory and experiment is very good despite the deliberate

naivete of the conception of the time-lag postulate.

The same nondimensionalization scheme as used in Chapter II is used here. (See Chapter I, page 13).

The equations of motion which describe the wave phenomenon may be transformed to characteristic coordinates and perturbed in a power series in some amplitude parameter ϵ as done in Chapter II. Again, the one-dimensional, unsteady equations of motion for a perfect gas in a homentropic flow field through a constant-area chamber yield the compatibility relations as follows:

$$\frac{2}{\gamma-1}\, da + du = 0 \qquad \text{along} \qquad \frac{dx}{dt} = u + a$$

$$\frac{2}{\gamma-1}\, da - du = 0 \qquad \text{along} \qquad \frac{dx}{dt} = u - a \tag{1}$$

Here, these equations apply to all orders in amplitude.

Each member of the family of characteristics with negative slope (in a t vs. x plot) has a particular value of the parameter α while each member of the family with positive slope has a particular value of the parameter β. α and β now become the independent variables and x and t become dependent variables such that

$$\begin{aligned} u &= u\,(\alpha,\beta) \\ a &= a\,(\alpha,\beta) \\ x &= x\,(\alpha,\beta) \\ t &= t\,(\alpha,\beta) \end{aligned}$$

Now, Equations (1) may be rewritten as partial differential equations

$$\frac{2}{\gamma-1}\frac{\partial a}{\partial \alpha}+\frac{\partial u}{\partial \alpha}=0$$

$$\frac{2}{\gamma-1}\frac{\partial a}{\partial \beta}-\frac{\partial u}{\partial \beta}=0$$

$$\frac{\partial x}{\partial \alpha}=(u+a)\frac{\partial t}{\partial \alpha}$$

$$\frac{\partial x}{\partial \beta}=(u-a)\frac{\partial t}{\partial \beta}$$

Writing the perturbation series, we have

$$u=u_0+\epsilon\, u_1(\alpha,\beta)+\epsilon^2 u_2(\alpha,\beta)+\epsilon^3 u_3(\alpha,\beta)+O(\epsilon^4)$$

$$a=1+\epsilon\, a_1(\alpha,\beta)+\epsilon^2 a_2(\alpha,\beta)+\epsilon^3 a_3(\alpha,\beta)+O(\epsilon^4)$$

$$p=1+\epsilon\, p_1(\alpha,\beta)+\epsilon^2 p_2(\alpha,\beta)+\epsilon^3 p_3(\alpha,\beta)+O(\epsilon^4)$$

$$x=x_0(\alpha,\beta)+\epsilon\, x_1(\alpha,\beta)+\epsilon^2 x_2(\alpha,\beta)+\epsilon^3 x_3(\alpha,\beta)+O(\epsilon^4)$$

$$t=t_0(\alpha,\beta)+\epsilon\, t_1(\alpha,\beta)+\epsilon^2 t_2(\alpha,\beta)+\epsilon^3 t_3(\alpha,\beta)+O(\epsilon^4) \qquad (2)$$

These series are substituted in the system of partial differential equations and the resulting system is separated according to powers in ϵ. The final set of equations is

u_0 = constant; a_0 = constant = 1

$$\frac{\partial x_0}{\partial \alpha}=(u_0+1)\frac{\partial t_0}{\partial \alpha}\ ;\ \frac{\partial x_0}{\partial \beta}=(u_0-1)\frac{\partial t_0}{\partial \beta} \qquad (3)$$

$$\frac{2}{\gamma-1}\frac{\partial a_i}{\partial \alpha}+\frac{\partial u_i}{\partial \alpha}=0\ ;\ \frac{2}{\gamma-1}\frac{\partial a_i}{\partial \beta}-\frac{\partial u_i}{\partial \beta}=0 \qquad (4)$$

where i = 1, 2, 3, etc.

$$\frac{\partial x_1}{\partial \alpha}=(u_0+1)\frac{\partial t_1}{\partial \alpha}+(u_1+a_1)\frac{\partial t_0}{\partial \alpha}$$

$$\frac{\partial x_1}{\partial \beta}=(u_0-1)\frac{\partial t_1}{\partial \beta}+(u_1-a_1)\frac{\partial t_0}{\partial \beta} \qquad (5)$$

$$\frac{\partial x_2}{\partial \alpha} = (u_0+1)\frac{\partial t_2}{\partial \alpha} + (u_1+a_1)\frac{\partial t_1}{\partial \alpha} + (u_2+a_2)\frac{\partial t_0}{\partial \alpha}$$

$$\frac{\partial x_2}{\partial \beta} = (u_0-1)\frac{\partial t_2}{\partial \beta} + (u_1-a_1)\frac{\partial t_1}{\partial \beta} + (u_2-a_2)\frac{d t_0}{\partial \beta} \qquad (6)$$

It is immediately shown that a_1, u_1, x_0 and t_0 are governed by homogeneous linear wave equations. If Equations (3) and (4) are differentiated and properly combined, the final forms obtained are

$$\frac{\partial^2 a_i}{\partial\alpha\partial\beta} = 0 \quad ; \quad \frac{\partial^2 u_i}{\partial\alpha\partial\beta} = 0$$

$$\frac{\partial^2 x_0}{\partial\alpha\partial\beta} = 0 \quad ; \quad \frac{\partial^2 t_0}{\partial\alpha\partial\beta} = 0$$

Similarly, it can be shown from Equations (5) and (6) that x_1, t_1, x_2, t_2, etc., are governed by inhomogeneous linear wave equations. If the analysis had been conducted in a space vs. time coordinate system, the coefficients a_i and u_i would have been governed by inhomogeneous wave equations whenever $i \geqslant 2$. This attainment of homogeneity is a most important simplification resulting from the transformation to characteristic coordinates.

The solution of the wave equations which are equivalent to Equations (3), (4), (5) and (6) requires the statement of initial conditions and boundary conditions. Concerning the solution for a_1 and u_1, a boundary condition will be given at the nozzle entrance and another boundary condition at the combustion zone. Initial conditions will be considered in an arbitrary functional form. Eventually, the initial condition will be replaced by the cyclic condition. The conditions on the solutions for x_o, t_o, x_1, t_1, etc. will come from the numbering system to be chosen for the characteristic coordinates.

The numbering system should be chosen in the most convenient manner. Since we are searching for solutions wherein the flow properties are single-valued, continuous functions (no shocks) of space and time, it is desirable to have a coordinate transformation which is single-valued and continuous. This will be different from the case of Chapter II where shocks were present (there the transformation was multi-valued). If care is not taken, multi-valueness of the transformation could be introduced by poor choice of the numbering system. This point will be demonstrated later.

As shown in Figure 8, the values of an α-characteristic and β-characteristic which intersect at $x = 0$ (combustion zone) are set equal. The value of an α-characteristic is taken to be greater than the value of a β-characteristic by one unit in their intersection at $x = 1$ (nozzle entrance). Also, at the initial time ($t=0$) the values of α and β of the characteristics intersecting at the point $x = 0$ are both taken to be zero. These result in the following conditions which will be applied to the solutions for x and t.

$$
\begin{aligned}
x &= 0 \qquad \text{at} \quad \alpha = \beta \\
x &= 1 \qquad \text{at} \quad \alpha = 1 + \beta \\
t &= 0 \qquad \text{at} \quad \alpha = \beta = 0
\end{aligned}
\tag{7}
$$

Note that here no distinction has been made between regions in the space vs. time plot and only one α, β sheet is considered. Of course, this differs from the analysis of Chapter II where different regions and different α, β sheets appear.

One more condition on the solutions for x and t is required. This is an initial condition and can essentially be reduced to a statement which gives the value of an α-characteristic at its intersection with the $\beta = 0$ characteristic (See Figure 8). A criterion for the single-valuedness of the transformation is that α be a monotonic function of the x position of the intersection with the $\beta = 0$ characteristic. In view of this criterion, a numbering system is chosen which states that to zero order, the number α is given by the value of x at the intersection with $\beta = 0$ characteristic. That is,

$$\alpha = x_0 \quad at \quad \beta = 0 \tag{8}$$

The higher order statement will be made later in an implicit form which allows a smooth and continuous transformation of the coordinates. ("Smooth" means analytic everywhere).

SOLUTION OF THE EQUATIONS

Equations (3), (4), (5), and (6) can now be solved. The solutions of Equations (4), (5), and (6) will be in terms of the arbitrary initial conditions as explained in the previous section. First, Equations (3) are rewritten in the form

$$\frac{\partial}{\partial \alpha}\left[\frac{x_0}{1+u_0} - t_0\right] = 0 \quad ; \quad \frac{\partial}{\partial \beta}\left[\frac{x_0}{1-u_0} + t_0\right] = 0$$

These possess the general solutions

$$\frac{x_0}{1+u_0} - t_0 = F_0(\beta) \; ; \; \frac{x_0}{1-u_0} + t_0 = G_0(\alpha) \tag{9}$$

Substitution for x and t from Equations (2) into Equations (7) and separation according to powers in ϵ yields conditions on the solutions of Equations (3), (5), and (6). The zero order results combined with Equation (8) provide the conditions necessary for the complete solution of Equation (3). These conditions are:

$$x_0 = 0 \quad \text{at} \quad \alpha = \beta$$
$$x_0 = 1 \quad \text{at} \quad \alpha = 1+\beta$$
$$t_0 = 0 \quad \text{at} \quad \alpha = \beta = 0$$
$$x_0 = \alpha \quad \text{at} \quad \beta = 0$$

The above conditions are sufficient to determine $G_0(\alpha)$ and $F_0(\beta)$ from Equations (9). The results are

$$G_0(\alpha) = \frac{2\alpha}{1-u_0^2} \quad ; \quad F_0(\beta) = -\frac{2\beta}{1-u_0^2}$$

Now, substituting for G_0 and F_0 in Equations (9), we find

$$x_0 = \alpha - \beta \quad ; \quad t_0 = \frac{\alpha}{1+u_0} + \frac{\beta}{1-u_0} \qquad (10)$$

It is readily seen that Equations ( ) have the general solutions

$$\frac{2}{\gamma-1} a_i + u_i = P_i(\beta) \; ; \; \frac{2}{\gamma-1} a_i - u_i = Q_i(\alpha)$$

Rewriting, we find

$$a_i = \frac{\gamma-1}{4}\left[P_i(\beta) + Q_i(\alpha)\right] \; ; \; u_i = \frac{P_i(\beta) - Q_i(\alpha)}{2} \qquad (11)$$

Note, of course, that $\frac{1}{2} P = \frac{1}{2}\sum_{i=0}^{\infty} \epsilon^i P_i$ and $\frac{1}{2} Q = \frac{1}{2}\sum_{i=0}^{\infty} \epsilon^i Q_i$ are Riemann invariants.

The coefficients P_i and Q_i could be determined from the boundary conditions and initial conditions on Equations (4). The boundary conditions are given by two relationships between the speed of sound perturbations and the gas velocity perturbations. One relationship is given at the combustion zone while the other is given at the nozzle entrance. Initial conditions would give the values of the function $Q_1(\alpha) = \frac{2}{\gamma-1} a_i - u_i$ along $\beta = 0$ in the range $0 \leq \alpha \leq 1$ and a_i along $\alpha = \beta$ in the range $-b_0 \leq \alpha \leq 0$ (or an equivalent statement). As developed in Appendix E, b_0 is the steady-state "time-lag" in characteristic coordinates. If there were no time-lag effects, the initial conditions would only be given along $\beta = 0$ in the range $0 \leq \alpha \leq 1$. This point will become clearer after the boundary condition at the combustion zone is applied. The solution will be found in terms of the arbitrary function $Q_1(\alpha)$ where $-b_0 \leq \alpha \leq 1$. The functions $P_1(\beta)$ and $Q_1(\alpha)$ where $\alpha > 1$ and $\beta \geq 0$ will be determined by means of the two boundary conditions. Later, the $Q_1(\alpha)$ for all α will be determined specifically so as to obtain periodic solutions.

The boundary condition for the short nozzle states that a wave is reflected at the nozzle with a loss in amplitude due to convection through the nozzle but with no phasing or dispersion. Whenever the length of the convergent portion of the nozzle is negligible compared to the wavelength, oscillatory processes may be considered as quasi-steady. This means that the Mach number at the nozzle entrance stays constant (since in the steady-state it depends on entrance to throat area ratio only). In other words, the gas velocity perturbation is directly proportional to the speed of sound perturbation. Use of Equations (2) for u and a leads to the boundary condition

$$u_i = u_0 a_i \quad \text{at} \quad \alpha = 1 + \beta$$

where $i = 0, 1\ 2$, etc. This condition may be written in a more convenient form by substitution from Equation (11). $Q_i(\alpha) = \frac{1-\nu}{1+\nu} P_i(\beta)$ at $\alpha = 1+\beta$ so the functions are related as shown by the following equation which is valid for all β.

$$Q_i(1+\beta) = \frac{1-\nu}{1+\nu} P_i(\beta) \tag{12}$$

Note the definition has been made $\nu = \frac{\gamma-1}{2} u_0$.

Since the coefficient $\frac{1-\nu}{1+\nu}$ is always less than unity and real, a loss in amplitude with no phase change in reflection at the nozzle is indicated.

The boundary condition at the other end of the chamber has been discussed in the previous section and is developed in Appendix E. If this boundary condition at the combustion zone as expressed by Equation (E-17) is combined with Equation (11), the result is written as follows:

$$\sum_{i=1}^{\infty} \epsilon^i \Big\{ \left[1+u_0(1-\gamma n)\right] P_i(\beta) - \left[1-u_0(1-\gamma n)\right] Q_i(\beta)$$
$$+ \gamma n u_0 \left[P_i(\beta - \delta_0) + Q_i(\beta - \delta_0) \right] \Big\} = 2u_0 \sum_{i=1}^{\infty} \epsilon^i R_i' \tag{13}$$

where the R_i' are defined in Appendix E. After separation according to powers in ϵ, the R_i' terms will appear as inhomogereous terms; i.e., they will not contain P_i and Q_i but will contain a_{i-1}, u_{i-1}, a_{i-2}, u_{i-2}, and so forth. These inhomogeneous terms depend upon lower order a_i and u_i and therefore, according to Equation (11), they depend upon lower order P_i and Q_i. Note, also, that the R_i' contain t_1 and t_2 so that the combustion zone boundary condition is related to the solutions of Equations (5) and (6). This dependence upon t_i is to higher order, so Equations (5) and (6) are not really coupled to Equation (13) after separation and the solutions can be found in an orderly manner.

Equation (13) describes the changes in amplitude and phase due to wave reflection at the combustion zone. At the combustion zone, energy is added to the oscillation by the combustion process. Since there is a finite time period associated with the combustion process (represented by the lag b_o in Equation (13)), a phasing appears in this energy addition process. On account of this phasing effect, there can either be an increase or a decrease in amplitude due to reflection at the combustion zone.

If the function $Q_i(\alpha) = \frac{2}{\gamma-1} a_i - u_i$ were given in the range $0 \leq \alpha \leq 1$ and if the function $a_1(\beta,\beta) = \frac{\gamma-1}{4}[P_i(\beta)+Q_i(\beta)]$ were given in the range $-b_o \leq \beta \leq 0$, Equation (13) could be used to solve for $P_1(\beta)$ in the range $0 \leq \beta \leq 1$. Then Equation (12) could be used to solve for $Q_1(\alpha)$ in the range $1 \leq \alpha \leq 2$. Returning to Equation (13), $P_1(\beta)$ would be determined for $1 \leq \beta \leq 2$. It follows that P_1 and Q_1 could be determined for all higher values of α and β by alternate use of Equations (12) and (13). (Note that R_1' is always considered as known since after separation, it is an inhomogeneous term which could be determined from lower order solutions). This above approach is not the one taken here. Instead, we shall seek $P_1(\beta)$ and $Q_1(\alpha)$ which satisfy Equations (12) and (13) and are continuous and smooth (analytic everywhere) functions of α and β. Periodic functions are the special type of these smooth functions in which we are most interested since these solutions represent equilibrium conditions.

Using Equation (12) to substitute for $P_1(\beta)$ in Equation (13), we have

$$\sum_{i=1}^{\infty} \epsilon^i \left\{ \frac{1+v}{1-v}\left[1+u_o(1-\gamma n)\right]Q_i(1+\beta) - \left[1-u_o(1-\gamma n)\right]Q_i(\beta) \right.$$

$$\left. + \gamma n\, u_o\left[\frac{1+v}{1-v}Q_i(1+\beta-b_o)+Q_i(\beta-b_o)\right]\right\} = 2u_o \sum_{i=1}^{\infty} \epsilon^i R_i' \qquad (14)$$

The above algebraic difference - equation is considered as applicable for all β ($-\infty \leq \beta \leq \infty$). A smooth function is sought as a solution of this equation for each coefficient Q_i. Once $Q_i(\alpha)$ is determined, then either Equation (12) or (13) may be used to solve for $P_i(\beta)$. Knowledge of both Q_i and P_i yields a_i and u_i through Equation (11). Before Equation (14) can be solved for $Q_i(\beta)$, R_i' must be determined for $i \geq 2$. These depend upon t_i as shown in Appendix E. So, prior to a solution of Equation (14) for Q_i where i = 2 or 3, Equations (5) and (6) must be solved for t_1 and t_2.

Using the solution for t_o as given by Equation (10), Equations (5) may be rewritten

$$\frac{\partial}{\partial\alpha}\left[\frac{x_1}{1+u_o} - t_1\right] = \frac{u_1+a_1}{(1+u_o)^2}$$

$$\frac{\partial}{\partial\beta}\left[\frac{x_1}{1-u_o} + t_1\right] = \frac{u_1-a_1}{(1-u_o)^2}$$

Substitution by means of Equation (11) has the following result

$$\frac{\partial}{\partial\alpha}\left[\frac{x_1}{1+u_0}-t_1\right]=\frac{\gamma+1}{4}\;\frac{P_1(\beta)}{(1+u_0)^2}-\frac{3-\gamma}{4}\;\frac{Q_1(\alpha)}{(1+u_0)^2}$$

$$\frac{\partial}{\partial\beta}\left[\frac{x_1}{1-u_0}+t_1\right]=\frac{3-\gamma}{4}\;\frac{P_1(\beta)}{(1-u_0)^2}-\frac{\gamma+1}{4}\;\frac{Q_1(\alpha)}{(1-u_0)^2}$$

These equations can be integrated to obtain

$$\frac{x_1}{1+u_0}-t_1=F_1(\beta)+\frac{\gamma+1}{4}\left(\frac{1}{1+u_0}\right)^2\alpha\,P_1(\beta)-\frac{3-\gamma}{4}\left(\frac{1}{1+u_0}\right)^2\int_0^{\alpha}Q_1(z)\,dz$$

$$\frac{x_1}{1-u_0}+t_1=G_1(\alpha)+\frac{3-\gamma}{4}\left(\frac{1}{1-u_0}\right)^2\int_0^{\beta}P_1(z)\,dz-\frac{\gamma+1}{4}\left(\frac{1}{1-u_0}\right)^2\beta\,Q_1(\alpha) \qquad (15)$$

where $F_1(\beta)$ and $G_1(\alpha)$ are the homogeneous solutions. The boundary conditions on x_1 and t_1 will be used to determine these functions. Substitution of Equation (2) into Equation (7) and separation of the first order terms yields the following boundary conditions

$$x_1=0 \quad \text{at} \quad \alpha=\beta$$
$$x_1=0 \quad \text{at} \quad \alpha=1+\beta$$
$$t_1=0 \quad \text{at} \quad \alpha=\beta=0 \qquad (16)$$

Application of the first condition on Equations (15) has the result

$$F_1(\beta)=-G_1(\beta)+\frac{3-\gamma}{4}\left[\frac{1}{(1-u_0)^2}\int_0^{\beta}P_1(z)\,dz-\frac{1}{(1+u_0)^2}\int_0^{\beta}Q_1(z)\,dz\right]$$
$$-\frac{\gamma+1}{4}\left[\left(\frac{1}{1+u_0}\right)^2P_1(\beta)-\left(\frac{1}{1-u_0}\right)^2Q_1(\beta)\right]\beta \qquad (17)$$

The second condition yields the result

$$G_1(1+\beta) = -F_1(\beta) + \frac{3-\gamma}{4}\left[\left(\frac{1}{1+u_0}\right)^2 \int_0^{1+\beta} Q_1(z)\,dz - \left(\frac{1}{1-u_0}\right)^2 \int_0^{\beta} P_1(z)\,dz\right]$$

$$+ \frac{\gamma+1}{4}\left[\left(\frac{1}{1-u_0}\right)^2 Q_1(1+\beta) - \left(\frac{1}{1+u_0}\right)^2 P_1(\beta)\right]\beta$$

$$+ \frac{\gamma+1}{4}\left(\frac{1}{1+u_0}\right)^2 P_1(\beta) \qquad (18)$$

Combination of the first and third condition yields the following

$$F_1(0) = G_1(0) = 0 \qquad (19)$$

If initial conditions were given such that the function $G_1(\alpha)$ were known in the range $0 \leq \alpha \leq 1$, Equation (17) could be used to calculate $F_1(\beta)$ in the range $0 \leq \beta \leq 1$. Then, Equation (18) could be used to determine $G_1(\alpha)$ in the range $1 \leq \alpha \leq 2$. Obviously, by successive alternate use of Equations (17) and (18), $F_1(\beta)$ and $G_1(\alpha)$ could be determined for all $\alpha > 1$ and all $\beta > 0$. P_1 and Q_1 would be calculated separately from Equation (12) and Equation (14) (after separation) and are considered as known in the above-mentioned calculations. Once F_1 and G_1 are known, Equations (15) yield x_1 and t_1.

The specification of $G_1(\alpha)$ for $0 \leq \alpha \leq 1$ essentially means the specification of the function $\frac{x_1}{1-u_0} + t_1$ along $\beta = 0$. It tells us to first order, when combined with Equations (2), (8), (15), and (17), where a characteristic with a given value of α intersects the $\beta = 0$ characteristic. This information is related in a pertinent manner to the numbering system for the characteristics since the inverse statement would give the value of α for the characteristic that intersects the $\beta = 0$ characteristic at a given x position. Therefore, the specification of $G_1(\alpha)$ in the range $0 \leq \alpha \leq 1$ is actually a specification of the numbering system.

The specification of $G_1(\alpha)$ must not violate the conditions (16). When these conditions are combined with Equations (15), it is seen that necessary conditions are

$$G_1(0) = 0; \qquad G_1(1) = t_1(1,0)$$

A straightforward specification will in general, violate these conditions. For example, $G_1(\alpha) = 0$ for $0 \leq \alpha \leq 1$ violates the second necessary condition. This can be shown to involve a double-valuedness of the transformation. The choice of numbering system such that $\alpha = x$ along $\beta = 0$ does not violate these conditions. (This would mean when combined with Equation (8) that $x_1 = 0$ along $\beta = 0$ and, therefore, by means of Equation (15), we see that $G_1(\alpha) = t_1(\alpha, 0)$).

The approach taken here will not involve the definite specification of the initial conditions; i.e., $G_1(\alpha)$ will be left arbitrary in the range $0 \leq \alpha \leq 1$. Instead, functions F_1 and G_1 will be sought which are very smooth for all α and β . This would introduce much simplification to the coordinate transformation. The same functional transformation would apply for all α and β . If $G_1(\alpha)$ were specified apriori in the range $0 \leq \alpha \leq 1$ (e.g., $\alpha = x$ along $\beta = 0$), then the functional form of the transformation would differ for different ranges of the values of α and β. In addition, and more importantly, it would be extremely difficult to determine whether a solution is periodic due to secular terms appearing in the transformation.

Subtraction of Equations (17) and (18) leads to the result

$$G_1(1+\beta) - G_1(\beta) = \frac{3-\gamma}{4}\left(\frac{1}{1+u_0}\right)^2 \int_{\beta}^{1+\beta} Q_1(z)\,dz +$$

$$+ \frac{\gamma+1}{4}\left\{\left(\frac{1}{1-u_0}\right)^2\left[Q_1(1+\beta) - Q_1(\beta)\right]\beta - \left(\frac{1}{1+u_0}\right)^2 P_1(\beta)\right\} \qquad (20)$$

If consideration is taken of Equations (12) of Chapter II, the above result agrees with Equation (16) of that chapter. The agreement exists because $x_1(\alpha,\beta)$ and $t_1(\alpha,\beta)$, if properly combined, are continuous through shock waves to first order.

Once P_1 and Q_1 are determined, the above algebraic-difference equation could be solved for G_1. Knowing this function, Equation (17) would yield F_1. If the functions G_1, P_1 and Q_1 were smooth, it is seen that the coordinate transformation functions $x(\alpha,\beta)$ and $t(\alpha,\beta)$ would be smooth to first order from Equations (10), (15) and (17). This allows the same transformation function to be applied for all values of α and β. In retrospect, it is seen that, in effect, the initial conditions for Equations (5) or, in other words, the coordinate numbering system was chosen such that $x(\alpha,\beta)$ and $t(\alpha,\beta)$ would be smooth functions to first order.

In order to solve for $G_1(\alpha)$, it is convenient to define a new function λ_1 such that

$$G_1(\alpha) = \lambda_1(\alpha) + \frac{\gamma+1}{4}\left(\frac{1}{1-u_0}\right)^2 Q_1(\alpha)\,\alpha +$$
$$+ \frac{3-\gamma}{4}\left(\frac{1}{1+u_0}\right)^2 \int_0^\alpha Q_1(z)\,dz \tag{21}$$

Note that since $G_1(0) = 0$ we see that $\lambda_1(0) \equiv 0$. Substitution of the above relation into Equation (20) and consideration of Equation (12) yields the result

$$\lambda_1(1+\beta) - \lambda_1(\beta) = -\frac{\gamma+1}{4}\left[\frac{1}{(1+u_0)^2}\,\frac{1+\nu}{1-\nu} + \left(\frac{1}{1-u_0}\right)^2\right] Q_1(1+\beta) \tag{22}$$

Substitution for F_1 and G_1 in Equations (15) by means of Equations (17) and (21) leads to the following

$$\frac{x_1}{1+u_0} - t_1 = -\lambda_1(\beta) + \frac{\gamma+1}{4}\,\frac{P_1(\beta)}{(1+u_0)^2}\,(\alpha-\beta) -$$
$$- \frac{3-\gamma}{4}\left[\frac{1}{(1+u_0)^2}\int_0^\alpha Q_1(z)\,dz + \left(\frac{1}{1-u_0}\right)^2\int_0^\beta P_1(z)\,dz\right]$$

$$\frac{x_1}{1-u_0} + t_1 = \lambda_1(\alpha) + \frac{\gamma+1}{4}\,\frac{Q_1(\alpha)}{(1-u_0)^2}\,(\alpha-\beta)$$
$$+ \frac{3-\gamma}{4}\left[\frac{1}{(1+u_0)^2}\int_0^\alpha Q_1(z)\,dz + \left(\frac{1}{1-u_0}\right)^2\int_0^\beta P_1(z)\,dz\right]$$

These two relations may be solved for x_1 and t_1 to obtain

$$x_1 = \frac{1-u_0^2}{2}\left[\lambda_1(\alpha)-\lambda_1(\beta)\right] + \frac{\gamma+1}{8}\left[\frac{1-u_0}{1+u_0}P_1(\beta) + \frac{1+u_0}{1-u_0}Q_1(\alpha)\right](\alpha-\beta) \tag{23a}$$

$$t_1 = \frac{1-u_0}{2}\lambda_1(\alpha) + \frac{1+u_0}{2}\lambda_1(\beta) + \frac{\gamma+1}{8}\left[\frac{Q_1(\alpha)}{1-u_0} - \frac{P_1(\beta)}{1+u_0}\right](\alpha-\beta)$$
$$+ \frac{3-\gamma}{4}\left[\left(\frac{1}{1+u_0}\right)^2\int_0^\alpha Q_1(z)\,dz + \left(\frac{1}{1-u_0}\right)^2\int_0^\beta P_1(z)\,dz\right] \tag{23b}$$

After $P_1(\beta)$ and $Q_1(\alpha)$ are determined from Equations (12) and (14), λ_1 may be determined from Equation (22) and then $x_1(\alpha,\beta)$ and $t_1(\alpha,\beta)$ may be determined from Equations (23).

Similar treatment could be given to all x_i and t_i, so that the coordinate transformation would be smooth to all orders. Considering x_2 and t_2, we may rewrite Equation (6) as follows:

$$\frac{\partial}{\partial\alpha}\left[\frac{x_2}{1+u_0} - t_2\right] = \frac{u_1+a_1}{1+u_0}\frac{\partial t_1}{\partial\alpha} + \frac{u_2+a_2}{1+u_0}\frac{\partial t_0}{\partial\alpha}$$

$$\frac{\partial}{\partial\beta}\left[\frac{x_2}{1-u_0} + t_2\right] = \frac{u_1-a_1}{1-u_0}\frac{\partial t_1}{\partial\beta} + \frac{u_2-a_2}{1-u_0}\frac{\partial t_0}{\partial\beta}$$

We substitute from Equations (10) and (11) to obtain

$$\frac{\partial}{\partial\alpha}\left[\frac{x_2}{1+u_0} - t_2\right] = \frac{1}{1+u_0}\left[\frac{\gamma+1}{4}P_1(\beta) - \frac{3-\gamma}{4}Q_1(\alpha)\right]\frac{\partial t_1}{\partial\alpha}$$
$$+ \left(\frac{1}{1+u_0}\right)^2\left[\frac{\gamma+1}{4}P_2(\beta) - \frac{3-\gamma}{4}Q_2(\alpha)\right] \tag{24a}$$

$$\frac{\partial}{\partial\beta}\left[\frac{x_2}{1-u_0} + t_2\right] = \frac{1}{1-u_0}\left[\frac{3-\gamma}{4}P_1(\beta) - \frac{\gamma+1}{4}Q_1(\alpha)\right]\frac{\partial t_1}{\partial\beta}$$
$$+ \left(\frac{1}{1-u_0}\right)^2\left[\frac{3-\gamma}{4}P_2(\beta) - \frac{\gamma+1}{4}Q_2(\alpha)\right] \tag{24b}$$

The derivatives $\frac{\partial t_1}{\partial \alpha}$ and $\frac{\partial t_1}{\partial \beta}$ are determined from Equation (23b) to be the following:

$$\frac{\partial t_1}{\partial \alpha} = \frac{1-u_0}{2} \frac{d\lambda_1}{d\alpha}(\alpha) + \frac{\gamma+1}{8}\left[\frac{Q_1(\alpha)}{1-u_0} - \frac{P_1(\beta)}{1+u_0}\right] +$$

$$+ \frac{\gamma+1}{8}(\alpha-\beta)\frac{1}{1-u_0}\frac{dQ_1}{d\alpha} + \frac{3-\gamma}{4}\left(\frac{1}{1+u_0}\right)^2 Q_1(\alpha)$$

$$\frac{\partial t_1}{\partial \beta} = \frac{1+u_0}{2}\frac{d\lambda_1}{d\beta}(\beta) - \frac{\gamma+1}{8}\left[\frac{Q_1(\alpha)}{1-u_0} - \frac{P_1(\beta)}{1+u_0}\right] -$$

$$- \frac{\gamma+1}{8}(\alpha-\beta)\frac{1}{1+u_0}\frac{dP_1}{d\beta} + \frac{3-\gamma}{4}\left(\frac{1}{1-u_0}\right)^2 P_1(\beta)$$

These two relations may be substituted into Equations (24a) and (24b). Integrating Equation (24a) and noting $\lambda_1(0) = 0$, we find

$$\frac{x_2}{1+u_0} - t_2 = F_2(\beta) + \frac{\gamma+1}{4}\frac{\alpha P_2(\beta)}{(1+u_0)^2} -$$

$$- \frac{3-\gamma}{4}\left(\frac{1}{1+u_0}\right)^2 \int_0^\alpha Q_2(z)dz + \frac{\gamma+1}{8}\left(\frac{1-u_0}{1+u_0}\right) P_1(\beta)\lambda_1(\alpha) -$$

$$- \frac{3-\gamma}{8}\frac{1-u_0}{1+u_0}\int_0^\alpha Q_1(z)\frac{d\lambda_1}{dz}(z)\,dz + \frac{(\gamma+1)(3-\gamma)(3+u_0)}{(1+u_0)^3} P_1(\beta)\int_0^\alpha Q_1(z)dz$$

$$- \frac{3-\gamma}{64}\left[\frac{\gamma+1}{1-u_0^2} + \frac{4(3-\gamma)}{(1+u_0)^3}\right]\int_0^\alpha Q_1^2(z)dz - \frac{(\gamma+1)^2}{32}\frac{1}{(1+u_0)^2} P_1^2(\beta)\alpha$$

$$+ \left[\frac{(\gamma+1)^2}{32}\frac{P_1(\beta)Q_1(\alpha)}{1-u_0^2} - \frac{(\gamma+1)(3-\gamma)}{64}\frac{Q_1^2(\alpha)}{1-u_0^2}\right](\alpha-\beta)$$

$$+ \left[\frac{(\gamma+1)^2}{32}\frac{P_1(\beta)Q_1(0)}{1-u_0^2} - \frac{(\gamma+1)(3-\gamma)}{64}\frac{Q_1^2(0)}{1-u_0^2}\right]\beta$$

(25a)

where $F_2(\beta)$ is the homogeneous solution. Integration of Equation (24b) yields the result:

$$\begin{aligned}
\frac{x_2}{1-u_0} + t_2 = {} & G_2(\alpha) + \frac{3-\gamma}{4}\left(\frac{1}{1-u_0}\right)^2 \int_0^\beta P_2(z)\,dz - \\
& - \frac{\gamma+1}{4}\,\frac{\beta Q_2(\alpha)}{(1-u_0)^2} + \frac{3-\gamma}{8}\,\frac{1+u_0}{1-u_0}\int_0^\beta P_1(z)\frac{d\lambda_1(z)}{dz}\,dz - \\
& - \frac{\gamma+1}{8}\,\frac{1+u_0}{1-u_0}\,Q_1(\alpha)\,\lambda_1(\beta) - \frac{(\gamma+1)(3-\gamma)(3-u_0)}{(1-u_0)^3}\,Q_1(\alpha)\int_0^\beta P_1(z)\,dz \\
& + \frac{3-\gamma}{16}\left[\frac{\gamma+1}{4}\,\frac{1}{1-u_0^2} + \frac{3-\gamma}{(1-u_0)^3}\right]\int_0^\beta P_1^2(z)\,dz + \frac{(\gamma+1)^2}{32}\,\frac{Q_1^2(\alpha)}{1-u_0^2}\,\beta \\
& + \frac{\gamma+1}{32}\,\frac{(\gamma+1)Q_1(\alpha)P_1(\beta) - \frac{3-\gamma}{2}P_1^2(\beta)}{1-u_0^2}(\alpha-\beta) + \\
& + \frac{\gamma+1}{32}\,\frac{P_1(0)}{1-u_0^2}\left[\frac{3-\gamma}{2}P_1(0) - (\gamma+1)\,Q_1(\alpha)\right]\alpha
\end{aligned} \tag{25b}$$

where $G_2(\alpha)$ is the homogeneous solution. Boundary conditions aid in the determination of the homogeneous solutions. Combination of Equations (2) and (7) and separation according to powers of ϵ results in the following boundary conditions

$$x_2 = 0 \quad \text{at} \quad \alpha = \beta$$

$$x_2 = 0 \quad \text{at} \quad \alpha = 1+\beta$$

$$t_2 = 0 \quad \text{at} \quad \alpha = \beta = 0$$

Application of the first condition to Equations (25a) and (25b) has the result

$$F_2(\beta) = -G_2(\beta) + \frac{\gamma+1}{4}\left[\frac{Q_2(\beta)}{(1-u_0)^2} - \frac{P_2(\beta)}{(1+u_0)^2}\right]\beta +$$

$$+\frac{3-\gamma}{4}\left[\left(\frac{1}{1+u_0}\right)^2 \int_0^\beta Q_2(z)dz - \left(\frac{1}{1-u_0}\right)^2 \int_0^\beta P_2(z)dz\right] +$$

$$+\frac{\gamma+1}{8}\left[\frac{1+u_0}{1-u_0} Q_1(\beta) - \frac{1-u_0}{1+u_0} P_1(\beta)\right]\lambda_1(\beta) + \frac{3-\gamma}{8}\int_0^\beta \left[\frac{1-u_0}{1+u_0} Q_1(z) - \frac{1+u_0}{1-u_0} P_1(z)\right]\frac{d\lambda_1}{dz}dz +$$

$$+(\gamma-1)(3-\gamma)\left[\frac{3-u_0}{(1-u_0)^3} Q_1(\beta)\int_0^\beta P_1(z)dz - \frac{3+u_0}{(1+u_0)^3} P_1(\beta)\int_0^\beta Q_1(z)dz\right] +$$

$$+\frac{3-\gamma}{16}\left\{\left[\frac{\gamma+1}{4}\frac{1}{1-u_0^2} + \frac{3-\gamma}{(1+u_0)^3}\right]\int_0^\beta Q_1^2(z)dz - \left[\frac{\gamma+1}{4}\frac{1}{1-u_0^2} + \frac{3-\gamma}{(1-u_0)^3}\right]\int_0^\beta P_1^2(z)dz\right\} +$$

$$+\frac{\gamma+1}{32}\left\{\frac{P_1(0)}{1-u_0^2}\left[(\gamma+1)Q_1(\beta) - \frac{3-\gamma}{2}P_1(0)\right] + \frac{\gamma+1}{(1+u_0)^2}P_1^2(\beta) +\right.$$

$$\left. + \frac{Q_1(0)}{1-u_0^2}\left[\frac{3-\gamma}{2}Q_1(0) - (\gamma+1)P_1(\beta)\right] - \frac{(\gamma+1)Q_1^2(\beta)}{1-u_0^2}\right\}\beta$$

(26)

Application of the second condition to Equations (25a) and (25b) yields

$$G_2(1+\beta) = -F_2(\beta) - \frac{\gamma+1}{4}\frac{P_2(\beta)}{(1+u_0)^2} + \frac{\gamma+1}{4}\left[\frac{Q_2(1+\beta)}{(1-u_0)^2} - \frac{P_2(\beta)}{(1+u_0)^2}\right]\beta$$

$$+\frac{3-\gamma}{4}\left[\frac{1}{(1+u_0)^2}\int_0^{1+\beta} Q_2(z)dz - \frac{1}{(1-u_0)^2}\int_0^\beta P_2(z)dz\right] +$$

$$+\frac{\gamma+1}{8}\left[\frac{1+u_0}{1-u_0}Q_1(1+\beta)\lambda_1(\beta) - \frac{1-u_0}{1+u_0}P_1(\beta)\lambda_1(1+\beta)\right] +$$

$$+ \frac{3-\gamma}{8}\left[\frac{1-u_0}{1+u_0}\int_0^{1+\beta} Q_1(z)\frac{d\lambda_1}{dz}(z)\,dz - \frac{1+u_0}{1-u_0}\int_0^{\beta} P_1(z)\frac{d\lambda_1}{dz}(z)\,dz\right] +$$

$$+ (\gamma+1)(3-\gamma)\left[\frac{3-u_0}{(1-u_0)^3} Q_1(1+\beta)\int_0^{\beta} P_1(z)dz - \frac{3+u_0}{(1+u_0)^3} P_1(\beta)\int_0^{1+\beta} Q_1(z)\,dz\right]$$

$$+ \frac{3-\gamma}{16}\left\{\left[\frac{\gamma+1}{4}\frac{1}{1-u_0^2} + \frac{3-\gamma}{(1+u_0)^3}\right]\int_0^{1+\beta} Q_1^2(z)dz - \left[\frac{\gamma+1}{4}\frac{1}{1-u_0^2} + \frac{3-\gamma}{(1-u_0)^3}\right]\int_0^{\beta} P_1^2(z)\,dz\right\}$$

$$+ \frac{\gamma+1}{32}\left\{\frac{P_1(0)}{1-u_0^2}\left[(\gamma+1)Q_1(1+\beta) - \frac{3-\gamma}{2}P_1(0)\right] + \frac{\gamma+1}{(1+u_0)^2}P_1^2(\beta) - \frac{\gamma+1}{1-u_0^2}Q_1^2(1+\beta)\right.$$

$$\left. -(\gamma+1)\frac{P_1(\beta)Q_1(0)}{1-u_0^2} + \frac{3-\gamma}{2}\frac{Q_1^2(0)}{1-u_0^2}\right\}\beta +$$

$$+ \frac{\gamma+1}{32}\left\{\frac{P_1(0)}{1-u_0^2}\left[(\gamma+1)Q_1(1+\beta) - \frac{3-\gamma}{2}P_1(0)\right] + \frac{\gamma+1}{(1+u_0)^2}P_1^2(\beta) +\right.$$

$$\left. + \frac{\frac{3-\gamma}{2}\left[P_1^2(\beta) + Q_1^2(1+\beta) - 2(\gamma+1)Q_1(1+\beta)P_1(\beta)\right]}{1-u_0^2}\right\} \qquad (27)$$

Combination of the first and third boundary conditions yields the result

$$F_2\ (0) = G_2\ (0) = 0 \qquad (28)$$

Subtracting Equation (26) from (27), we find

$$G_2(1+\beta) - G_2(\beta) = \frac{\gamma+1}{4}\left\{\left[\frac{Q_2(1+\beta) - Q_2(\beta)}{(1-u_0)^2}\right]\beta - \frac{P_2(\beta)}{(1+u_0)^2}\right\} +$$

$$+ \frac{3-\gamma}{4}\left(\frac{1}{1+u_0}\right)^2 \int_\beta^{1+\beta} Q_2(z)dz + \frac{\gamma+1}{8}\left\{\frac{1+u_0}{1-u_0}\left[Q_1(1+\beta) - Q_1(\beta)\right]\lambda_1(\beta) -\right.$$

$$\left. - \frac{1-u_0}{1+u_0}\left[\lambda_1(1+\beta) - \lambda_1(\beta)\right]P_1(\beta)\right\} + \frac{3-\gamma}{8}\left(\frac{1-u_0}{1+u_0}\right)\int_\beta^{1+\beta} Q_1(z)\frac{d\lambda_1}{dz}(z)dz +$$

$$+ (\gamma+1)(3-\gamma)\left\{\frac{3-u_0}{(1-u_0)^3}\left[Q_1(1+\beta) - Q_1(\beta)\right]\int_0^\beta P_1(z)dz - \frac{3+u_0}{(1+u_0)^3}P_1(\beta)\int_\beta^{1+\beta} Q_1(z)dz\right\}$$

$$+ \frac{3-\gamma}{16}\left[\frac{\gamma+1}{4}\;\frac{1}{1-u_0^2} + \frac{3-\gamma}{(1+u_0)^3}\right]\int_\beta^{1+\beta} Q_1^2(z)\,dz +$$

$$+ \frac{(\gamma+1)^2}{32(1-u_0^2)}\left\{P_1(0)\left[Q_1(1+\beta) - Q_1(\beta)\right] + Q_1^2(\beta) - Q_1^2(1+\beta)\right\}\beta$$

$$+ \frac{\gamma+1}{32}\left\{\frac{(\gamma+1)}{(1+u_0)^2}P_1^2(\beta) + \frac{\frac{3-\gamma}{2}\left[P_1^2(\beta) + Q_1^2(1+\beta) - P_1^2(0)\right]}{1-u_0^2} +\right.$$

$$+ \frac{(\gamma+1)\,Q_1(1+\beta)\left[P_1(0) - 2P_1(\beta)\right]}{1-u_0^2} \qquad (29)$$

The same approach is used for the solution of $x_2(\alpha,\beta)$ and $t_2(\alpha,\beta)$ as was used for the solution of $x_1(\alpha,\beta)$ and and $t_1(\alpha,\beta)$. Instead of specifying initial conditions along $\beta = 0$, the above equation is solved for $G_2(\beta)$. Once this is known, F_2, x_2, and t_2 may be found from Equations

(25a), (25b), and (26).

In solving for $G_2(\beta)$, it is convenient to define a new function $\lambda_2(\beta)$ such that

$$
\begin{aligned}
G_2(\beta) = {} & \lambda_2(\beta) + \frac{\gamma+1}{4}\frac{1}{(1-u_0)^2}\,\beta Q_2(\beta) + \frac{3-\gamma}{4}\frac{1}{(1+u_0)^2}\int_0^\beta Q_2(z)\,dz \\
& + \frac{\gamma+1}{8}\,\frac{1+u_0}{1-u_0}\,Q_1(\beta)\lambda_1(\beta) + \frac{3-\gamma}{8}\,\frac{1-u_0}{1+u_0}\int_0^\beta Q_1(z)\frac{d\lambda_1}{dz}(z)\,dz + \\
& + \frac{3-\gamma}{16}\left[\frac{\gamma+1}{4}\left(\frac{1}{1-u_0^2}\right) + \frac{3-\gamma}{(1+u_0)^3}\right]\int_0^\beta Q_1^2(z)\,dz + \frac{(\gamma+1)^2}{32}\,\frac{P_1(0)Q_1(\beta)}{1-u_0^2}\,\beta \\
& - \frac{(\gamma+1)^2}{32}\,\frac{1}{1-u_0^2}\,Q_1^2(\beta)\beta - \frac{\gamma+1}{32}\left(\frac{3-\gamma}{2}\right)\frac{P_1^2(0)}{1-u_0^2}\,\beta + \\
& + (\gamma+1)(3-\gamma)\,\frac{3-u_0}{(1-u_0)^3}\,Q_1(\beta)\int_0^\beta P_1(z)\,dz
\end{aligned}
\tag{30}
$$

Since $G_2(0)=0$ and $\lambda_1(0)=0$, it follows that $\lambda_2(0)=0$. Substitution of the above relation into Equation (29) and use of Equations (12) and (22) to substitute for $P_1(\beta)$ and $[\lambda_1(1+\beta)-\lambda_1(\beta)]$ leads to the following equation for λ_2

$$
\begin{aligned}
\lambda_2(1+\beta) - \lambda_2(\beta) = {} & r_1\,Q_2(1+\beta) + r_2\,Q_1^2(1+\beta) \\
& + r_3\,Q_1(1+\beta)\int_\beta^{1+\beta} P_1(z)\,dz + r_4\,Q_1(1+\beta)\int_\beta^{1+\beta} Q_1(z)\,dz
\end{aligned}
\tag{31}
$$

where the definitions are made

$$K = \frac{1+\nu}{1-\nu}$$

$$r_1 \equiv -\frac{\gamma+1}{4}\left[\frac{K}{(1+u_0)^2} + \left(\frac{1}{1-u_0}\right)^2\right]$$

$$r_2 \equiv \frac{(\gamma+1)^2}{32}\left\{\left[K\,\frac{1-u_0}{1+u_0} + \frac{1+u_0}{1-u_0}\right]\left[\frac{K}{(1+u_0)^2} + \frac{1}{(1-u_0)^2}\right] + \right.$$

$$\left. + \frac{1-\left[2K - \frac{3-\gamma}{2(\gamma+1)}(K^2-1)\right]}{1-u_0^2} + \left(\frac{K}{1+u_0}\right)^2\right\}$$

$$r_3 \equiv -(\gamma+1)(3-\gamma)\,\frac{3-u_0}{(1-u_0)^3}$$

$$r_4 \equiv -(\gamma+1)(3-\gamma)\,\frac{3+u_0}{(1+u_0)^3}\,K$$

Substitution for $F_2(\beta)$ in Equation (25a) by means of Equations (26) and (30) has the result

$$\frac{x_2}{1+u_0} - t_2 = -\lambda_2(\beta) + \frac{\gamma+1}{4}\,\frac{P_2(\beta)}{(1+u_0)^2}(\alpha-\beta) -$$

$$-\frac{3-\gamma}{4}\left[\frac{1}{(1+u_0)^2}\int_0^\alpha Q_2(z)\,dz + \frac{1}{(1-u_0)^2}\int_0^\beta P_2(z)\,dz\right]$$

$$+\frac{\gamma+1}{8}\,\frac{1-u_0}{1+u_0}\,P_1(\beta)\left[\lambda_1(\alpha)-\lambda_1(\beta)\right] - \frac{3-\gamma}{8}\left[\frac{1-u_0}{1+u_0}\int_0^\alpha Q_1(z)\frac{d\lambda_1}{dz}\,dz\right. +$$

$$\left.+\frac{1+u_0}{1-u_0}\int_0^\beta P_1(z)\frac{d\lambda_1}{dz}\,dz\right] + (\gamma+1)(3-\gamma)\,\frac{3+u_0}{(1+u_0)^3}\,P_1(\beta)\int_\beta^\alpha Q_1(z)\,dz -$$

$$-\frac{3-\gamma}{16}\left\{\left[\frac{\gamma+1}{4(1-u_0^2)} + \frac{3-\gamma}{(1+u_0)^3}\right]\int_0^\alpha Q_1^2(z)\,dz + \right.$$

$$\left. + \left[\frac{\gamma+1}{4(1-u_0^2)} + \frac{3-\gamma}{(1-u_0)^3}\right]\int_0^\beta P_1^2(z)\,dz\right\} +$$

$$+\frac{\gamma+1}{32}\left\{\frac{(\gamma+1)Q_1(\alpha)\,P_1(\beta)}{1-u_0^2} - \frac{3-\gamma}{2}\,\frac{Q_1^2(\alpha)}{1-u_0^2} - \frac{(\gamma+1)P_1^2(\beta)}{(1+u_0)^2}\right\}(\alpha-\beta)$$

Equation (30) is used to substitute for $G_2(\alpha)$ in Equation (25b) with the final result:

$$\frac{x_2}{1-u_0}+t_2=\lambda_2(\alpha)+\frac{\gamma+1}{4}\,\frac{Q_2(\alpha)}{(1-u_0)^2}(\alpha-\beta)+$$

$$+\frac{3-\gamma}{4}\left\{\left(\frac{1}{1-u_0}\right)^2\int_0^\beta P_2(z)\,dz+\left(\frac{1}{1+u_0}\right)^2\int_0^\alpha Q_2(z)\,dz\right\}+$$

$$+\frac{3-\gamma}{8}\left\{\frac{1+u_0}{1-u_0}\int_0^\beta P_1(z)\frac{d\lambda_1}{dz}\,dz+\frac{1-u_0}{1+u_0}\int_0^\alpha Q_1(z)\frac{d\lambda_1}{dz}\,dz\right\}+$$

$$+\frac{\gamma+1}{8}\,\frac{1+u_0}{1-u_0}\,Q_1(\alpha)\left[\lambda_1(\alpha)-\lambda_1(\beta)\right]+\frac{(\gamma+1)(3-\gamma)(3-u_0)}{(1-u_0)^3}\,Q_1(\alpha)\int_\beta^\alpha P_1(z)\,dz$$

$$+\frac{3-\gamma}{16}\left\{\left[\frac{\gamma+1}{4}\,\frac{1}{1-u_0^2}+\frac{3-\gamma}{(1-u_0)^3}\right]\int_0^\beta P_1^2(z)\,dz+\left[\frac{\gamma+1}{4}\,\frac{1}{1-u_0^2}+\frac{3-\gamma}{(1+u_0)^3}\right]\int_0^\alpha Q_1^2(z)\,dz\right\}$$

$$+\frac{\gamma+1}{32}\left\{\frac{(\gamma+1)Q_1(\alpha)\left[P_1(\beta)-Q_1(\alpha)\right]-\frac{3-\gamma}{2}P_1^2(\beta)}{1-u_0^2}\right\}(\alpha-\beta)$$

These two relations may be solved for x_2 and t_2. The result for x_2 is as follows:

$$x_2 = \frac{1-u_0^2}{2}\left[\lambda_2(\alpha)-\lambda_2(\beta)\right] + \frac{\gamma+1}{8}\left[\frac{1-u_0}{1+u_0}P_2(\beta) + \frac{1+u_0}{1-u_0}Q_2(\alpha)\right](\alpha-\beta)$$

$$+ \frac{\gamma+1}{16}\left[\lambda_1(\alpha)-\lambda_1(\beta)\right]\left[(1-u_0)^2 P_1(\beta) + (1+u_0)^2 Q_1(\alpha)\right] +$$

$$+ \frac{(\gamma+1)(3-\gamma)}{2}\left\{\frac{(3+u_0)(1-u_0)}{(1+u_0)^2}P_1(\beta)\int_\beta^\alpha Q_1(z)\,dz + \frac{(3-u_0)(1+u_0)}{(1-u_0)^2}Q_1(\alpha)\int_\beta^\alpha P_1(z)\,dz\right\}$$

$$+ \frac{\gamma+1}{64}\left\{2(\gamma+1)Q_1(\alpha)P_1(\beta) - \left(\frac{5+\gamma}{2}\right)Q_1^2(\alpha) - \left[(\gamma+1)\frac{1-u_0}{1+u_0} + \frac{3-\gamma}{2}\right]P_1^2(\beta)\right\}(\alpha-\beta) \tag{32a}$$

The following result is found for t_2

$$t_2 = \frac{1-u_0}{2}\lambda_2(\alpha) + \frac{1+u_0}{2}\lambda_2(\beta) + \frac{\gamma+1}{8}\left[\frac{Q_2(\alpha)}{1-u_0} - \frac{P_2(\beta)}{1+u_0}\right](\alpha-\beta)$$

$$+ \frac{3-\gamma}{4}\left[\left(\frac{1}{1+u_0}\right)^2\int_0^\alpha Q_2(z)\,dz + \left(\frac{1}{1-u_0}\right)^2\int_0^\beta P_2(z)\,dz\right] +$$

$$+ \frac{\gamma+1}{16}\left[\lambda_1(\alpha)-\lambda_1(\beta)\right]\left[(1+u_0)Q_1(\alpha) - (1-u_0)P_1(\beta)\right] +$$

$$+ \frac{3-\gamma}{8}\left[\frac{1-u_0}{1+u_0}\int_0^\alpha Q_1(z)\frac{d\lambda_1}{dz}\,dz + \frac{1+u_0}{1-u_0}\int_0^\beta P_1(z)\frac{d\lambda_1}{dz}\,dz\right] +$$

$$+ \frac{(\gamma+1)(3-\gamma)}{2}\left\{\frac{3-u_0}{(1-u_0)^2}Q_1(\alpha)\int_\beta^\alpha P_1(z)\,dz - \frac{3+u_0}{(1+u_0)^2}P_1(\beta)\int_\beta^\alpha Q_1(z)\,dz\right\}$$

$$+ \frac{3-\gamma}{16}\left\{\left[\frac{\gamma+1}{4(1-u^2)} + \frac{3-\gamma}{(1-u_0)^3}\right]\int^\beta P_1^2(z)\,dz + \left[\frac{\gamma+1}{4(1-u^2)} + \frac{3-\gamma}{(1+u_0)^3}\right]\int^\alpha Q_1^2(z)\,dz\right\}$$

$$+ \frac{\gamma+1}{32}\left\{\left[\frac{3-\gamma}{4}\,\frac{1}{1-u_0} - \frac{\gamma+1}{2}\left(\frac{1}{1+u_0}\right)\right]Q_1^2(\alpha) - \frac{(\gamma+1)u_0 Q_1(\alpha) P_1(\beta)}{1-u_0^2} + \right.$$

$$\left. + \frac{3\gamma-1}{4(1+u_0)} P_1^2(\beta)\right\}(\alpha \cdot \beta) \tag{32b}$$

Once P_1, Q_1, P_2, and Q_2 are determined, Equations (22) and (31) yield λ_1 and λ_2 and then Equations (32a) and (32b) yield $x_2(\alpha,\beta)$ and $t_2(\alpha,\beta)$.

FIRST ORDER SOLUTION

A first order solution will be found for Equation (14). After separation, the first order coefficients of that equation are related in the following manner

$$K[1+u_0(1-\gamma n)]Q_1(1+\beta) - [1-u_0(1-\gamma n)]Q_1(\beta) +$$

$$+ \gamma n u_0 \; K \; Q_1(1+\beta-b_0) + \gamma n \; u_0 \; Q_1(\beta-b_0) = 0 \tag{33}$$

If it is noted that r and s are constants and i is the imaginary unit, a solution of the form

$$Q_1(\beta) = \tfrac{1}{2}\left[e^{(r+is)\beta} + e^{(r-is)\beta}\right]$$

is found to satisfy Equation (33) if n, b_0 (or $\tilde{\zeta}_0$), s, and r are related in a specific manner. (Note that γ and u_0 are parameters in this relationship). The relationships are found by means of substitution for Q_1 in Equation (33) to be as follows:

$$K[1+u_0(1-\gamma n)]e^{r\pm is} - [1-u_0(1-\gamma n)] +$$

$$+ \gamma n u_0 \, K e^{r(1-b_0)} \, e^{\pm is(1-b_0)} + \gamma n u_0 \, e^{-rb_0} \, e^{\mp isb_0} = 0 \tag{34}$$

Consideration of the upper or lower sign yields identical results so that only the upper sign is used here.

We are primarily interested in periodic solutions which result whenever $r = 0$. If Q_1 is periodic, Equation (11) and (12) indicate that P_1, u_1, and a_1 also are periodic. In the analysis of periodic phenomena, it is convenient to transform to a coordinate system where the period is equal to 2π. The rules of the transformation for the present case would be

$$\psi_\alpha \equiv s\alpha \quad ; \quad \psi_\beta \equiv s\beta \quad ; \quad \phi \equiv s b_0 \tag{35}$$

In this new coordinate system, the period is known, but the transformation parameter s has to be determined. This parameter s is actually the angular frequency in the old coordinate system. The frequency will be a function of ϵ in the old coordinate system but is always 2π in the new coordinate system which produces a good deal of simplification. In essence, this transformation is identical to the one performed in Appendix A.

Setting $r = 0$ and transforming coordinates, we obtain the following from Equation (34)

$$K\left[1 + u_0(1 - \gamma n)\right]e^{is} - \left[1 - u_0(1 - \gamma n)\right] + \gamma n u_0 K e^{i(s-\phi)} + \gamma n u_0 e^{-i\phi} = 0 \tag{36}$$

The above complex relationship represents (after separation of real and imaginary parts) two relationships between s, ϕ, and n. The values which satisfy these relationships are denoted by $s^{(o)}$, $\phi^{(o)}$, and $n^{(o)}$. These are the values of s, ϕ, and n which produce periodic solutions or neutral oscillations for infinitesimally small perturbations since Equation (33) gives the asymptotic behavior of Equation (14) as the amplitude parameter goes to zero. These two relationships are

$$K[1+u_o(1-\gamma n^{(o)})]\cos s^{(o)} - [1-u_o(1-\gamma n^{(o)})] + \\ + \gamma n^{(o)} u_o K \cos(s^{(o)} - \phi^{(o)}) + \gamma n^{(o)} u_o \cos\phi^{(o)} = 0$$

$$K[1+u_o(1-\gamma n^{(o)})]\sin s^{(o)} + \gamma n^{(o)} u_o K \sin(s^{(o)} - \phi^{(o)}) \\ - \gamma n^{(o)} u_o \sin\phi^{(o)} = 0$$

If terms of the order $\frac{\gamma-1}{2} u_o^2$ are considered as negligible compared to unity the above relationships* simplify to the following forms

$$\sin\phi^{(o)} = \frac{\tan\frac{1}{2}s^{(o)}}{\gamma n^{(o)} u_o}$$

$$\cos\phi^{(o)} = 1 - \frac{\gamma+1}{2\gamma n^{(o)}}$$

Neglecting terms of the order u_o^2, the approximate solutions are found to be

$$s^{(o)} = 2\ell\pi \pm 2\gamma n^{(o)} u_o \left[1-\left(1-\frac{\gamma+1}{2\gamma n^{(o)}}\right)^2\right]^{\frac{1}{2}}$$

$$\phi^{(o)} = (2m+1)\pi \mp \left[\frac{\pi}{2} - \arc\sin\left(\frac{\gamma+1}{2\gamma n^{(o)}} - 1\right)\right] \qquad (37)$$

*These relationships have the exact solution

$$\cos s^{(o)} = \frac{\{1+u_o^2[(1-\gamma n^{(o)})^2 - (\gamma n^{(o)})^2]\}(K^2+1) + 2u_o(1-\gamma n^{(o)})(K^2-1)}{2K\{1-u_o^2[(1-\gamma n^{(o)})^2 - (\gamma n^{(o)})^2]\}}$$

$$\cos\phi^{(o)} = -\frac{\{1+u_o^2[(1-\gamma n^{(o)})^2 + (\gamma n^{(o)})^2]\}(K^2-1) - 2u_o(1-\gamma n^{(o)})(K^2+1)}{2\gamma n^{(o)} u_o[K^2+1+u_o(1-\gamma n^{(o)})(K^2-1)]}$$

If u_o is not so small, these exact solutions should be used instead of the approximate solutions.

m and ℓ are positive integers (0, 1, 2, etc). ℓ indicates the approximate number of half-wavelengths contained in the chamber length. That is, for the fundamental mode $\ell = 1$, for the second harmonic $\ell = 2$, and so forth. The possible frequencies of oscillation equal the natural resonant chamber frequencies plus (or minus) a small correction (of the order of the Mach number) due to the time-lag. The resonant frequencies only occur whenever $n^{(0)} = \frac{\gamma+1}{4\gamma}$. Since $\phi^{(0)}$ is a product of the lag $b_0^{(0)}$ and the frequency $s^{(0)}$, it represents the ratio of the time-lag to the period of oscillation. Therefore, it is seen from Equation (37) that the neutral oscillation occurs whenever the time-lag is approximately an odd multiple of one-half the period. It is also seen that $s^{(0)}$ and $\phi^{(0)}$ are double-valued functions of $n^{(0)}$.

If terms of the order of u_o^2 are neglected, one can readily see by tracing back through the transformations that

$$s^{(0)} = 2\,\omega^{(0)} \qquad \phi^{(0)} = \omega^{(0)}\tau_0^{(0)}$$

where ω is the angular frequency in time coordinates. Substitution into Equation (37) yields the result:

$$\omega^{(0)} = \ell\pi \pm \gamma n^{(0)} u_o \left[1 - \left(1 - \frac{\gamma+1}{2\gamma n^{(0)}}\right)^2\right]^{1/2}$$

$$\omega^{(0)}\tau_0^{(0)} = (2m+1)\pi \mp \left[\frac{\pi}{2} - \arcsin\left(\frac{\gamma+1}{2\gamma n^{(0)}} - 1\right)\right] \qquad (38)$$

These are the identical results found by Crocco through a small perturbation analysis. This is perfectly understandable since as $\epsilon \rightarrow 0$, the perturbation becomes infinitesimally small. However, whenever ϵ is not infinitesimally small, the relationships between s (or ω), ϕ (or τ_o), and n will differ. It is the purpose of this analysis to find the relationship between s, ϕ, n, and ϵ. This will result from analysis of the nonlinear terms. Only to lowest order in ϵ, will the results of that analysis agree with those found by Crocco since modifications occur when finite-amplitude oscillations are considered. It is important to note that there is a "continuity" between the results of Crocco's small perturbation analysis and this nonlinear analysis as indicated by the identity when the amplitude parameter goes to zero. The nonlinear modifications to Crocco's results will be shown to be order ϵ or higher.

As can be seen from Equations (37) or (38), the chamber may oscillate at various frequencies depending upon the values of $\phi^{(0)}$(or $\tau_o^{(0)}$) and $n^{(0)}$. Figure 9 shows curves of $\tau_o^{(0)}$ vs. $n^{(0)}$ for various modes of oscillation. A stability analysis (see Ref. 1) shows that a small perturbation grows (unstable) on the shaded side of these curves while it decays (stable) on the other side. As already mentioned, the small perturbation would remain the same size along this line. Figure 9 shows wide ranges of the τ_o vs. n plot where only one mode of oscillation occurs; e.g., if $\tau_o = 1$ and n slightly greater than $\frac{\gamma+1}{4\gamma}$, only the fundamental modes occur. None of the higher harmonics are expected (to this order in ϵ). The results reported in Ref. 2 show that the ranges of τ_o and n where this non-superposition occurs are the ranges of practical interest. Therefore, our nonlinear analysis will be simplified by assuming that to lowest order in ϵ only one harmonic appears. All other modes are stable. The amplitude of that harmonic which appears is defined as ϵ. This gives specific meaning to this parameter which until the present has had only a vague meaning. The oscillating part of $Q(\alpha)$ is now $\epsilon \cos \psi_\alpha$ plus terms of order ϵ^2.

Given $n^{(o)}$, $s^{(o)}$ may be determined from Equation (37). Now, $Q_1(\psi_\alpha)$ is known and Equations (11) and (12) yield u_1, a_1, and P_1. The results are

$$Q_1(\psi_\alpha) = \frac{1}{2}\left[e^{i\psi_\alpha} + e^{-i\psi_\alpha}\right]; \quad P_1(\psi_\beta) = \frac{K}{2}\left[e^{is}e^{i\psi_\beta} + e^{-is}e^{-i\psi_\beta}\right]$$

$$u_1 = \frac{1}{4}\left[Ke^{is}e^{i\psi_\beta} - e^{i\psi_\alpha} + Ke^{-is}e^{-i\psi_\beta} - e^{-i\psi_\alpha}\right]$$

$$a_1 = \frac{\gamma-1}{8}\left[Ke^{is}e^{i\psi_\beta} + e^{i\psi_\alpha} + Ke^{-is}e^{-i\psi_\beta} + e^{-i\psi_\alpha}\right] \tag{39}$$

Note that as points in the ζ, n plane are considered which are slightly distant from the $\zeta_0^{(o)}$ vs. $n^{(o)}$ curve, s becomes different from $s^{(o)}$ in the above relations. Furthermore, ϵ may be different from zero so that the higher order terms become important.

Next x_1 and t_1 will be determined. Substituting from Equation (39) for $Q_1(1+\beta)$ on the right-hand side of Equation (22) we have

$$\lambda_1(s+\psi_\beta) - \lambda_1(\psi_\beta) = -\frac{\gamma+1}{8}\left[\frac{K}{(1+u_0)^2} + \left(\frac{1}{1-u_0}\right)^2\right]\left\{e^{is}e^{i\psi_\beta} + e^{-is}e^{-i\psi_\beta}\right\}$$

The solution is readily found to be of the following form (if the condition $\lambda_1(0) = 0$ is noted)

$$\lambda_1(\psi_\beta) = K_1\left(e^{i\psi_\beta} - 1\right) + K_1^*\left(e^{-i\psi_\beta} - 1\right) \tag{40}$$

where the following definitions are made for the complex conjugates K_1 and K_1^*

$$K_1 \equiv -\frac{\gamma+1}{16}\,\frac{1-e^{-is}}{1-\cos s}\left[\frac{K}{(1+u_0)^2} + \frac{1}{(1-u_0)^2}\right]$$

$$K_1^* \equiv -\frac{\gamma+1}{16}\,\frac{1-e^{is}}{1-\cos s}\left[\frac{K}{(1+u_0)^2} + \frac{1}{(1-u_0)^2}\right]$$

Note that the solution to the homogeneous part has been omitted. Any function with period s may be added to the above solution and Equation (22) is still satisfied. As is seen from Equation (21), this function would appear in the same additive manner in the solution for G_1. This means therefore that many specifications of $G_1(\alpha)$ in the range $0 \leq \alpha \leq 1$ are possible which would give a smooth coordinate transformation for all α and β. We shall consider the simplest one by setting the homogeneous solution to Equation (22) equal to zero.

Equation (37) shows that as $n^{(0)} \to \frac{\gamma+1}{4\gamma}$, $s^{(0)} \to 2\ell\pi$. Therefore, both K and K_1^* go to infinity as the resonant frequencies are approached. This solution is not valid whenever $s = 2\pi$. Returning to Equation (22) and setting $s = 2\pi$, we have

$$\lambda_1(1+\beta) - \lambda_1(\beta) = -\frac{\gamma+1}{8}\left[\frac{K}{(1+u_0)^2} + \frac{1}{(1-u_0)^2}\right]\left\{e^{2\pi i\beta} + e^{-2\pi i\beta}\right\}$$

The forcing function on the right-hand side has the period 1. This is also the period of the homogeneous solution ($\lambda_1(1+\beta)=\lambda_1(\beta)$) so that the infinite amplitudes of the periodic solution are due to the forcing function having the resonant frequency. The proper solution in this case is not periodic but instead is given by the following

$$\lambda_1(\beta) = -\frac{\gamma+1}{8}\left[\frac{K}{(1+u_0)^2} + \frac{1}{(1-u_0)^2}\right]\left\{e^{2\pi i\beta} + e^{-2\pi i\beta}\right\}\beta$$

This means that corrections on the location of the characteristics in x, t space oscillate with the same frequency as the flow field except near the resonant frequency. Near the resonant frequency a secular solution is found which indicates continual distortion and may be interpreted as meaning that a shock must eventually form before the solution can be periodic.

For our purposes, it shall be assumed that the frequency is sufficiently far from the resonant one and the amplitude is sufficiently small to obtain periodic solutions without shock waves. If these were not so, e.g., ϵ large, double-valuedness of the coordinate transformation might occur indicating shock formation.

Using Equations (39) and (40) to substitute in Equation (23a), we obtain the result

$$x_1 = \frac{\gamma+1}{16}\left\{\left[K\,\frac{1-u_0}{1+u_0} + \frac{1+u_0}{1-u_0}\right]\left[\frac{e^{is}\left(e^{i\psi_\beta}-e^{i\psi_\alpha}\right)}{e^{is}-1} + \frac{e^{-is}\left(e^{-i\psi_\beta}-e^{-i\psi_\alpha}\right)}{e^{-is}-1}\right]\right.$$
$$\left. + \left[\frac{1+u_0}{1-u_0}\left(e^{i\psi_\alpha}+e^{-i\psi_\alpha}\right) + K\,\frac{1-u_0}{1+u_0}\left(e^{is}e^{i\psi_\beta}+e^{-is}e^{-i\psi_\beta}\right)\right]\left(\frac{\psi_\alpha-\psi_\beta}{s}\right)\right\}$$

(41a)

Note that consideration of the transformation (35) has been taken. Similar substitution in Equation (23b) has the result:

$$t_1 = \frac{\gamma+1}{16}\left\{\left[K\,\frac{1-u_0}{1+u_0} + \frac{1+u_0}{1-u_0}\right]\frac{e^{is}}{e^{is}-1}\left[\frac{1-e^{i\psi_\alpha}}{1+u_0} + \frac{1-e^{i\psi_\beta}}{1-u_0}\right]\right.$$
$$\left. + \left[\frac{e^{i\psi_\alpha}}{1-u_0} - K\,\frac{e^{is}}{1+u_0}\,e^{i\psi_\beta}\right]\left(\frac{\psi_\alpha-\psi_\beta}{s}\right)\right\} +$$
$$- \frac{3-\gamma}{8}\,\frac{i}{s}\left[K\,\frac{e^{is}}{(1-u_0)^2}\left(e^{i\psi_\beta}-1\right) + \frac{1}{(1+u_0)^2}\left(e^{i\psi_\alpha}-1\right)\right]$$
$$+ \frac{\gamma+1}{16}\left\{\left[K\,\frac{1-u_0}{1+u_0} + \frac{1+u_0}{1-u_0}\right]\frac{e^{-is}}{e^{-is}-1}\left[\frac{1-e^{-i\psi_\alpha}}{1+u_0} + \frac{1-e^{-i\psi_\beta}}{1-u_0}\right]\right.$$
$$\left. + \left[\frac{e^{-i\psi_\alpha}}{1-u_0} - K\,\frac{e^{-is}}{1+u_0}\,e^{-i\psi_\beta}\right]\left(\frac{\psi_\alpha-\psi_\beta}{s}\right)\right\}$$
$$+ \frac{3-\gamma}{8}\,\frac{i}{s}\left[K\,\frac{e^{-is}}{(1-u_0)^2}\left(e^{-i\psi_\beta}-1\right) + \frac{1}{(1+u_0)^2}\left(e^{-i\psi_\alpha}-1\right)\right] \tag{41b}$$

Note from Equations (39), (41a), and (41b) that P_1, Q_1, u_1, a_1, x_1, and t_1 are equal to a complex function plus its conjugate. They are actually real quantities but are left in this form for the purpose of simplifying operations which involve these quantities.

From Equations (2), (10), and (35), it is seen that

$$\frac{\psi_\alpha - \psi_\beta}{s} = \alpha - \beta = \psi - \epsilon\psi_1 + O(\epsilon^2)$$

$$\frac{\alpha}{1+u_0} + \frac{\beta}{1-u_0} = t - \epsilon t_1 + O(\epsilon^2)$$

α and β may be found to be the following

$$\frac{\psi_\alpha}{s} = \alpha = \frac{1}{2}\left[(1+u_0)x + (1-u_0^2)t\right] -$$

$$-\frac{\epsilon}{2}\left[(1+u_0)x_1 + (1-u_0^2)t_1\right] + O(\epsilon^2)$$

$$\frac{\psi_\beta}{s} = \beta = \frac{1}{2}\left[(1-u_0^2)t - (1-u_0)x\right] -$$

$$-\frac{\epsilon}{2}\left[(1-u_0^2)t_1 - (1-u_0)x_1\right] + O(\epsilon^2)$$

Substitution of the above statements into Equations (41a) and (41b) for ψ_α, ψ_β and $\psi_\alpha - \psi_\beta$ readily shows that x_1 and t_1 are periodic functions in time at fixed space positions, that is, neglecting terms of order ϵ^2, a displacement (in time vs. space plot) equal to the period of oscillation (x held constant) results in a certain definite change in α and β. The change is the same for both α and β and equals the period in characteristic coordinates. These statements are true regardless of the space position or the initial time. The implication is that there is no convergence of the characteristics (shock formation) provided that ϵx_1 and ϵt_1 are not so large as to cause double-valuedness of the transformation. Note that if the oscillation were at the resonant frequency, the periodic solution would not be obtained.

SECOND ORDER SOLUTION

The primary interest is in finite amplitude oscillations where higher order effects are important. In general, these oscillations occur at values of ϕ (or ζ_0), n, and s(or ω) other than those values at the neutral stability line($\phi^{(0)}$, $n^{(0)}$, $s^{(0)}$)

We shall be concerned with values of these parameters close to those values on the neutral line. The amplitude of the oscillation will be assumed to depend in some continuous manner on the displacement from the neutral line in a parameter plot (e.g., τ_o vs. n). This will be implied in a convenient manner by writing the parameters as a series in ϵ about the neutral line value. The series are written as follows:

$$\begin{aligned}
\phi &= \phi^{(0)} + \epsilon\,\phi^{(1)} + \epsilon^2\,\phi^{(2)} + ---- \\
b_o &= b_o^{(0)} + \epsilon\, b_o^{(1)} + \epsilon^2\, b_o^{(2)} + ---- \\
\tau_o &= \tau_o^{(0)} + \epsilon\,\tau_o^{(1)} + \epsilon^2\,\tau_o^{(2)} + ---- \\
n &= n^{(0)} + \epsilon\, n^{(1)} + \epsilon^2\, n^{(2)} + ---- \\
s &= s^{(0)} + \epsilon\, s^{(1)} + \epsilon^2\, s^{(2)} + ---- \\
\omega &= \omega^{(0)} + \epsilon\,\omega^{(1)} + \epsilon^2\,\omega^{(2)} + ----
\end{aligned} \tag{42}$$

This of course, is analogous to the eigenvalue perturbations of Appendix A. Here, the parameters are not eigenvalues in the strict sense because they appear in the boundary condition (combustion zone) and not in the differential equations. It is also analogous to the approach of Chapter II where the amplitude was assumed to depend upon the displacement from the neutral stability line (actually a point since only one parameter ω appears).

The transformation (35) is applied to Equation (14) and the series (42) are used to substitute* for ϕ, n, and s.

* A Taylor series expansion is used whenever the series appears in the argument of a function. For example, $Q_i(s+\psi_\beta) = Q_i(s^{(0)}+\psi_\beta) + \frac{dQ_i}{d\psi_\beta}(s^{(0)}+\psi_\beta)\left[\epsilon s^{(1)} + \epsilon^2 s^{(2)}\right] + \frac{1}{2}\frac{d^2Q_i}{d\psi_\beta^2}(s^{(0)}+\psi_\beta)\left[\epsilon s^{(1)}\right]^2 + o(\epsilon^3)$

Now all orders may be separated according to powers in ϵ. The second order equation is written as follows:

$$K\left[1+u_o(1-\gamma n^{(o)})\right]Q_2(s^{(o)}+\psi_\beta)-\left[1-u_o(1-\gamma n^{(o)})\right]Q_2(\psi_\beta)$$
$$+\gamma n^{(o)}u_o\left[KQ_2(s^{(o)}+\psi_\beta-\phi^{(o)})+Q_2(\psi_\beta-\phi^{(o)})\right]=2u_oR_2 \qquad (43)$$

where the definition is made

$$R_2=R_2'-\Big[-KQ_1(s^{(o)}+\psi_\beta)-Q_1(\psi_\beta)+KQ_1(s^{(o)}+\psi_\beta-\phi^{(o)})+$$
$$+Q_1(\psi_\beta-\phi^{(o)})\Big]\frac{\gamma n^{(1)}}{2}-K\Big\{\left[1+u_o(1-\gamma n^{(o)})\right]\frac{dQ_1}{d\psi_\beta}(s^{(o)}+\psi_\beta)+$$
$$+\gamma n^{(o)}u_o\frac{dQ_1}{d\psi_\beta}(s^{(o)}+\psi_\beta-\phi^{(o)})\Big\}\frac{s^{(1)}}{2u_o}+\Big\{K\frac{dQ_1}{d\psi_\beta}(s^{(o)}+\psi_\beta-\phi^{(o)})+$$
$$+\frac{dQ_1}{d\psi_\beta}(\psi_\beta-\phi^{(o)})\Big\}\frac{\gamma n^{(o)}}{2}\phi^{(1)}$$

First, R_2' is evaluated from the first order results. Equations (39) are used to evaluate a_1 at the point ψ_β, ψ_β (or β, β) and at the point $\psi_\beta-\phi^{(o)}, \psi_\beta-\phi^{(o)}$ (or $\beta-b_o^{(o)}, \beta-b_o^{(o)}$) and u_1 at ψ_β, ψ_β. Equation (41b) is used to evaluate t_1 at these same points. The result of these substitutions in Equation (E-17) is the following

$$R_2'=(a_r+ia_i)e^{2i\psi_\beta}+(a_r-ia_i)e^{-2i\psi_\beta}+c_r \qquad (44)$$

where the following definitions apply

$$a_r\equiv g_1\left[K^2\cos 2s+2K\cos s+1\right]+$$
$$+g_2\left[K^2\cos 2(s-\phi)+2K\cos(s-2\phi)+\cos 2\phi\right]+$$

$$+ q_3\left[K^2\cos(2s-\phi)+2K\cos(s-\phi)+\cos\phi\right]$$

$$+ q_4\Big[K^2\big(\cos 2(s-\phi)-\cos(2s-\phi)\big)+$$

$$+2K\big(\cos(s-2\phi)-\cos(s-\phi)\big)+\cos 2\phi-\cos\phi\Big]+$$

$$+\frac{n\gamma(\gamma+1)s\left[K\frac{1-u_0}{1+u_0}+\frac{1+u_0}{1-u_0}\right]}{128\,(1-\cos s)}\Big[\sin(s+\phi)-\sin s-\sin\phi+$$

$$+K\big(\sin\phi-\sin s+\sin(s-\phi)\big)\Big]-$$

$$-\frac{n\gamma(3-\gamma)}{64}\,K\,\frac{1+u_0}{1-u_0}\Big[\cos s-\cos(s-\phi)+K\big(\cos 2s-\cos(2s-\phi)\big)\Big]$$

$$-\frac{n\gamma(3-\gamma)}{64}\,\frac{1-u_0}{1+u_0}\Big[1-\cos\phi+K\big(\cos s-\cos(s-\phi)\big)\Big]$$

$$a_i \equiv q_1\left[K^2\sin 2s+2K\sin s\right]+$$

$$+ q_2\left[K^2\sin 2(s-\phi)+2K\sin(s-2\phi)-\sin 2\phi\right]$$

$$+ q_3\left[K^2\sin(2s-\phi)+2K\sin(s-\phi)-\sin\phi\right]$$

$$+ q_4\Big[K^2\big(\sin 2(s-\phi)-\sin(2s-\phi)\big)+$$

$$+2K\big(\sin(s-2\phi)-\sin(s-\phi)\big)+\sin\phi-\sin 2\phi\Big]+$$

$$+\frac{n\gamma(\gamma+1)s}{128(1-\cos s)}\left[K\frac{1-u_0}{1+u_0}+\frac{1+u_0}{1-u_0}\right]\Big[1+\cos(s+\phi)-$$

$$-\cos s-\cos\phi+K\big(\cos s-\cos(s-\phi)+\cos\phi-1\big)\Big]-$$

$$- \frac{n\gamma(3-\gamma)}{64} \frac{1-u_0}{1+u_0} \left[K\left(\sin s - \sin(s-\phi)\right) + \sin\phi\right] -$$

$$- \frac{n\gamma(3-\gamma)}{64} K \frac{1+u_0}{1-u_0} \left[\sin s - \sin(s-\phi) + K\left(\sin 2s - \sin(2s-\phi)\right)\right]$$

$$c_r \equiv 2\left[g_1 + g_2 + g_3 \cos\phi + g_4(\cos\phi - 1)\right]\left[K^2 +\right.$$

$$\left. + 1 + 2K\cos s\right] -$$

$$- \frac{n\gamma(\gamma+1)s}{64(1-\cos s)} \left[K \frac{1-u_0}{1+u_0} + \frac{1+u_0}{1-u_0}\right] \left\{(K-1)\left[\sin s - \sin(s+\phi)\right] -\right.$$

$$\left. - \sin\phi + K\left[\sin(2s+\phi) - \sin 2s\right]\right\} +$$

$$+ \frac{n\gamma(3-\gamma)}{32} \frac{1-u_0}{1+u_0} \left\{1 - \cos\phi + K\left[\cos s - \cos(s+\phi)\right]\right\}$$

$$+ \frac{n\gamma(3-\gamma)}{32} K \frac{1+u_0}{1-u_0} \left\{\cos s - \cos(s+\phi) + K\left[\cos 2s - \cos(2s+\phi)\right]\right\}$$

$$g_1 \equiv \left(\frac{\gamma-1}{8}\right)^2 M - \frac{\gamma n}{16} + \frac{\gamma+1}{64}$$

$$g_2 \equiv \left(\frac{\gamma n}{4}\right)^2 - \left(\frac{\gamma-1}{8}\right)^2 M$$

$$g_3 \equiv \frac{\gamma n(1-\gamma n)}{16}$$

$$g_4 \equiv \left(\frac{\gamma n}{4}\right)^2$$

It is understood in the above relationships that s represents $s^{(0)}$, ϕ represents $\phi^{(0)}$, and n represents $n^{(0)}$. In order to be consistent the inhomogeneous term should be evaluated with zero order parameters. Any error would be of higher order and should be added unto the inhomogeneous terms of the higher order equations.

Combination of Equations (39) and (44) and the definition following Equation (43) yields the result

$$
\begin{aligned}
R_2 = {} & (a_r + i a_i) e^{2i\psi_\beta} + (a_r - i a_i) e^{-2i\psi_\beta} + \\
& + c_r - \Big\{ \Big[-K e^{is^{(0)}} - 1 + K e^{i(s^{(0)} - \phi^{(0)})} + e^{-i\phi^{(0)}} \Big] \frac{e^{i\psi_\beta}}{2} \\
& + \Big[-K e^{-is^{(0)}} - 1 + K e^{-i(s^{(0)} - \phi^{(0)})} + e^{+i\phi^{(0)}} \Big] \frac{e^{-i\psi_\beta}}{2} \Big\} \frac{\gamma n^{(1)}}{2} \\
& - \Big\{ iK \Big(\big[1 + u_0 (1 - \gamma n^{(0)}) \big] e^{is^{(0)}} + \gamma n^{(0)} u_0 \, e^{i(s^{(0)} - \phi^{(0)})} \Big) \frac{e^{i\psi_\beta}}{2} - \\
& - iK \Big(\big[1 + u_0 (1 - \gamma n^{(0)}) \big] e^{-is^{(0)}} + \gamma n^{(0)} u_0 \, e^{-i(s^{(0)} - \phi^{(0)})} \Big) \frac{e^{-i\psi_\beta}}{2} \Big\} \frac{s^{(1)}}{2u_0} \\
& + \Big\{ i \Big[K e^{i(s^{(0)} - \phi^{(0)})} + e^{-i\phi^{(0)}} \Big] \frac{e^{i\psi_\beta}}{2} - \\
& \qquad - i \Big[K e^{-i(s^{(0)} - \phi^{(0)})} + e^{i\phi^{(0)}} \Big] \frac{e^{-i\psi_\beta}}{2} \Big\} \frac{\gamma n^{(0)}}{2} \phi^{(1)}
\end{aligned}
\tag{45}
$$

Now Q_2 must be a function such that whenever operated upon in the manner indicated by Equation (43), another function will be produced which satisfies Equation (45). Since ϵ is defined as the amplitude of the lowest harmonic, this lowest harmonic can be contained only in the Q_1 term and not in the Q_2 term. This lowest harmonic ($e^{i\psi/\beta}$, $e^{-i\psi/\beta}$) is referred to as the first harmonic even though lower values of s might be possible if n and ζ_0 were in different ranges. With this definition of ϵ, Q_2 contains only a constant plus the second harmonic ($e^{2i\psi/\beta}$, $e^{-2i\psi/\beta}$).

The difficulty of the appearance of the first harmonic in the inhomogeneous part of Equation (43) is overcome by setting its coefficient equal to zero. This gives a complex linear homogeneous relation (or two real relations) between $n^{(1)}$, $s^{(1)}$, and $\phi^{(1)}$. These linear homogeneous relations are trivial however since they only allow displacements tangent to the zero order curve in a ϕ, n, s plot. (This is analoguous to the m = 2 case in Appendix A). No generality is lost by setting $n^{(1)} = \phi^{(1)} = s^{(1)} = 0$. This means that the perturbations of n, ϕ (or ζ_0), s (or ω) will be of second (or higher) order.

With this trivial result for the first order perturbations of the parameters, it is seen that R_2 and R_2' are identical. Q_2 may contain only a constant term and a second harmonic term. Assume it is of the form

$$Q_2(\psi) = (A_r + i A_i)e^{2i\psi} + (A_r - iA_i)e^{-2i\psi} + C_r \qquad (46)$$

Use of Equations (44) and (46) to substitute in Equation (43) yields the result

$$(\eta_r + i\eta_i)(A_r + iA_i)e^{2i\psi} + (\eta_r - i\eta_i)(A_r - iA_i)e^{-2i\psi}$$

$$+ \frac{(\gamma+1)u_0}{1-\nu} C_r = 2u_0 R_2 = 2u_0 (a_r + ia_i)e^{2i\psi} +$$

$$+ 2u_0 (a_r - ia_i)e^{-2i\psi} + 2u_0 c_r$$

where the following definitions have been made

$$\eta_r \equiv K\left[1 + u_0(1-\gamma n^{(0)})\right]\cos 2s^{(0)} -$$

$$-\left[1 - u_0(1-\gamma n^{(0)})\right] + \gamma n^{(0)} u_0 \left[K\cos 2(s^{(0)} - \phi^{(0)}) + \cos 2\phi^{(0)}\right]$$

$$\eta_i \equiv K\left[1 + u_0(1-\gamma n^{(0)})\right]\sin 2s^{(0)} + \gamma n^{(0)} u_0 \left[K\sin 2(s^{(0)} - \phi^{(0)}) -\right.$$

$$\left. - \sin 2\phi^{(0)}\right]$$

Separation of the above equation according to coefficients of $e^{2i\psi}$, $e^{-2i\psi}$ and 1 yields the three equations

$$(\eta_r + i\eta_i)(A_r + iA_i) = 2u_0 (a_r + ia_i)$$

$$(\eta_r - i\eta_i)(A_r - iA_i) = 2u_0 (a_r - ia_i)$$

$$\frac{(\gamma+1)u_0}{1-\nu} C_r = 2u_0 c_r$$

The first and second relation are actually identical and yield the same results. Each is complex however and actually two relations which may be used to solve for A_r and A_i. The third relation is used to solve for C_r . (a_r, a_i, and c_r are known from lower order analysis). The final results are

$$A_r = 2u_o \left(\frac{\eta_i a_i + \eta_r a_r}{\eta_r^2 + \eta_i^2} \right)$$

$$A_i = 2u_o \left(\frac{\eta_r a_i - \eta_i a_r}{\eta_r^2 + \eta_i^2} \right)$$

$$C_r = \frac{2(1-\nu)}{(\gamma+1)} c_r \tag{47a}$$

If the definitions are made

$$\Theta \equiv \frac{1}{2} \text{ arc tan} \left(\frac{A_i}{A_r} \right)$$

$$A \equiv \frac{A_r}{\cos 2\Theta} = \frac{A_i}{\sin 2\Theta} \tag{47b}$$

Equation (46) simplifies to the following form

$$Q_2(\psi_\alpha) = 2A \cos 2(\psi_\alpha + \Theta) + C_r \tag{48a}$$

Substitution in Equation (12) and use of Equations (11) yields the results

$$P_2(\psi_\beta) = 2AK \cos 2(\psi_\beta + s + \Theta) + K C_r$$

$$a_2 = \frac{\gamma-1}{2} A \left[\cos 2(\psi_\alpha + \Theta) + K \cos 2(\psi_\beta + s + \Theta) \right] + \frac{\gamma-1}{4}(1+K) C_r$$

$$u_2 = A \left[K \cos 2(\psi_\beta + s + \Theta) - \cos 2(\psi_\alpha + \Theta) \right] + \frac{1}{2}(K-1) C_r \tag{48b}$$

In general, $s \neq s^{(o)}$ in the above relations but the difference is of order ϵ^2. Since these are second order coefficients given above, the effect of the difference is of order ϵ^4 which will be seen to be negligible since the analysis will not include fourth order terms. So, for our purposes, $s = s^{(o)}$ in the above relationships. (This is not so in Equations (39) where the difference produces a third order effect).

Equation (48) show that the second order effects include the addition of a second harmonic to the waveform of the oscillation plus a correction to the mean pressure and velocity in the combustion chamber. There is a phase between the first and second harmonic terms represented by θ. The amplitude of the second harmonic term is proportional to $\epsilon^2 A$ and the magnitudes of the corrections to the mean flow conditions are proportional to $\epsilon^2 C_r$. (There will be an additional correction for the mean pressure produced by a $\epsilon^2 a_1^2$ term as will later be seen).

Now, x_2 and t_2 can be determined by solving for λ_2 from Equation (31). Equations (39) and (48) are used to substitute in that equation (with consideration given to the transformation (35). Then Equation (31) is rewritten in the following manner:

$$\lambda_2(s+\psi) - \lambda_2(\psi) = \left\{ r_1 C_r + \frac{r_2}{2} + \frac{r_3 K}{2s} \sin s + \frac{r_4}{2s} \sin s \right\}$$
$$+ \left\{ r_1 A e^{2i(s+\theta)} + \frac{r_2}{4} e^{2is} - \frac{r_3}{4}\left(e^{3is} - e^{2is}\right)\frac{Ki}{s} - \right.$$
$$\left. - \frac{r_4}{4}\frac{i}{s}\left(e^{2is} - e^{is}\right)\right\} e^{2i\psi} + \left\{ r_1 A e^{-2i(s+\theta)} + \right.$$
$$\left. + \frac{r_2}{4} e^{-2is} + \frac{r_3}{4}\left(e^{-3is} - e^{-2is}\right)\frac{Ki}{s} + \frac{r_4}{4}\frac{i}{s}\left(e^{-2is} - e^{-is}\right)\right\} e^{-2i\psi}$$

with the condition $\lambda_2(0) = 0$. The solution is found to be the following

$$\lambda_2(\psi) = V_1\left(e^{2i\psi}-1\right) + V_1^*\left(e^{-2i\psi}-1\right) + V_2\frac{2\psi}{s} \tag{49}$$

with the following definitions

$$V_1 \equiv \frac{1}{2(1-\cos 2s)}\Big[r_1 A\left(e^{2i\theta} - e^{2i(s+\theta)}\right) + \frac{r_2}{4}\left(1-e^{2is}\right)$$
$$- \frac{r_3 Ki}{4s}\left(e^{is} - 1 - e^{3is} + e^{2is}\right) - \frac{r_4 i}{4s}\left(1 - e^{-is} - e^{2is} + e^{is}\right)\Big]$$

$$V_1^* \equiv \frac{1}{2(1-\cos 2s)}\Big[r_1 A\left(e^{-2i\theta} - e^{-2i(s+\theta)}\right) + \frac{r_2}{4}\left(1-e^{-2is}\right) +$$
$$+ \frac{r_3 Ki}{4s}\left(e^{is} - 1 - e^{-3is} + e^{-2is}\right) + \frac{r_4 i}{4s}\left(1 - e^{is} - e^{-2is} + e^{-is}\right)\Big]$$

$$V_2 \equiv \left\{r_1 C_r + \frac{r_2}{2} + r_3\frac{K\sin s}{2s} + \frac{r_4}{2s}\sin s\right\}$$

Note that v_1 and v_1^* are conjugate.

In similar fashion to the solution for λ_1 , the homogeneous solution has been omitted for the sake of simplicity. The addition of the homogeneous solution would mean a change in the coordinate numbering system as can be seen from Equation (30).

Note that the solution for λ_2 given by (49) goes to infinity as $s \longrightarrow 2\ell\pi$ where ℓ is an integer. This periodic solution is invalid near the resonant frequency in the same manner that the periodic solution for λ_1 is invalid.

Now Equations (32a) and (32b) will yield x_2 and t_2

after certain substitutions. Equations (39), (40), (48), and (49) give P_1, Q_1, λ_1, P_2, Q_2, and λ_2. The associated differentiation and integration operations are tedious but straightforward so they are not given in detail here. Only the result for t_2 is presented here:

$$t_2 = \frac{1-u_0}{2} V_2 \frac{\psi_\alpha}{s} + \frac{1+u_0}{2} V_2 \frac{\psi_\beta}{s} + \frac{\gamma+1}{8}\left[\frac{1}{1-u_0} - \frac{K}{1+u_0}\right] C_r \frac{\psi_\alpha - \psi_\beta}{s} +$$

$$+ \frac{3-\gamma}{4}\left(\frac{1}{1+u_0}\right)^2 C_r \frac{\psi_\alpha}{s} + \frac{3-\gamma}{4}\frac{1}{(1-u_0)^2} K C_r \frac{\psi_\beta}{s} +$$

$$+ \frac{\gamma+1}{16}\left[\frac{1+u_0}{2}(K_1 + K_1^*) + \frac{1-u_0}{2} K (K_1 e^{-is} + K_1^* e^{is})\right] -$$

$$- \frac{\gamma+1}{16}\left[\left(\frac{1+u_0}{2} K_1^* + \frac{1-u_0}{2} K K_1 e^{-is}\right) e^{i(\psi_\alpha - \psi_\beta)} +\right.$$

$$\left. + \left(\frac{1+u_0}{2} K_1 + \frac{1-u_0}{2} K K_1^* e^{is}\right) e^{-i(\psi_\alpha - \psi_\beta)}\right] +$$

$$+ \frac{3-\gamma}{16}\,\frac{1-u_0}{1+u_0}\, i(K_1 - K_1^*)\psi_\alpha + \frac{3-\gamma}{16}\,\frac{1+u_0}{1-u_0}\, K i (K_1 e^{-is} - K_1^* e^{is})\psi_\beta$$

$$- \frac{(\gamma+1)(3-\gamma)(3-u_0)}{8(1-u_0)^2}\,\frac{Ki}{s}\left[e^{is} - e^{-is} + e^{-is} e^{i(\psi_\alpha - \psi_\beta)} - e^{is} e^{-i(\psi_\alpha - \psi_\beta)}\right]$$

$$+ \frac{(\gamma+1)(3-\gamma)(3+u_0)}{8(1+u_0)^2}\,\frac{Ki}{s}\left[e^{is} - e^{-is} + e^{-is} e^{i(\psi_\alpha - \psi_\beta)} - e^{is} e^{-i(\psi_\alpha - \psi_\beta)}\right]$$

$$+ \frac{3-\gamma}{16}\left[\frac{\gamma+1}{4(1-u_0^2)} + \frac{3-\gamma}{(1-u_0)^3}\right]\frac{K^2}{2}\,\frac{\psi_\beta}{s} + \frac{3-\gamma}{16}\left[\frac{\gamma+1}{4(1-u_0^2)} + \frac{3-\gamma}{(1+u_0)^3}\right]\frac{\psi_\alpha}{2s} +$$

$$+ \left(\frac{\gamma+1}{32}\right)\left(\frac{\psi_\alpha - \psi_\beta}{s}\right)\left\{-(\gamma+1)\,\frac{u_0}{1-u_0^2}\,\frac{K}{4}\left(e^{-is} e^{i(\psi_\alpha - \psi_\beta)} + e^{is} e^{-i(\psi_\alpha - \psi_\beta)}\right) +\right.$$

$$\left. + \frac{1}{2}\left[\frac{3-\gamma}{4}\,\frac{1}{1-u_0} - \frac{\gamma+1}{2}\,\frac{1}{1+u_0}\right] + \frac{3\gamma - 1}{8}\,\frac{1}{1+u_0}\right\} +$$

$$+\frac{1-u_0}{2}V_1\left(e^{2i\psi_\alpha}-1\right)+\frac{1+u_0}{2}V_1\left(e^{2i\psi_\beta}-1\right)+\frac{\gamma+1}{8s}\frac{Ae^{2i\theta}e^{2i\psi_\alpha}}{1-u_0}\left(\psi_\alpha-\psi_\beta\right)$$

$$-\frac{\gamma+1}{8s}\frac{KAe^{2i(s+\theta)}e^{2i\psi_\beta}}{1+u_0}\left(\psi_\alpha-\psi_\beta\right)-\frac{3-\gamma}{8s}\frac{iAe^{2i\theta}}{(1+u_0)^2}\left(e^{2i\psi_\alpha}-1\right)$$

$$-\frac{3-\gamma}{8s}\frac{iAKe^{2i(\theta+s)}}{(1-u_0)^2}\left(e^{2i\psi_\beta}-1\right)\quad+\frac{\gamma+1}{32}(1+u_0)K_1e^{2i\psi_\alpha}+$$

$$+\frac{\gamma+1}{32}(1-u_0)KK_1e^{is}e^{2i\psi_\beta}-\frac{\gamma+1}{32}\left[(1+u_0)K_1+(1-u_0)KK_1e^{is}\right]e^{i(\psi_\alpha+\psi_\beta)}$$

$$+\frac{3-\gamma}{32}\left[\frac{1-u_0}{1+u_0}K_1\left(e^{2i\psi_\alpha}-1\right)+\frac{1+u_0}{1-u_0}KK_1e^{is}\left(e^{2i\psi_\beta}-1\right)\right]$$

$$-\frac{(\gamma+1)(3-\gamma)(3-u_0)}{8(1-u_0)^2}\frac{Ki}{s}e^{is}\left(e^{2i\psi_\alpha}-e^{i(\psi_\alpha+\psi_\beta)}\right)+$$

$$+\frac{(\gamma+1)(3-\gamma)(3+u_0)}{8(1+u_0)^2}\frac{Ki}{s}e^{is}\left(e^{2i\psi_\beta}-e^{i(\psi_\alpha+\psi_\beta)}\right)-$$

$$-\frac{3-\gamma}{16}\left[\frac{\gamma+1}{4(1-u_0^2)}+\frac{3-\gamma}{(1-u_0)^3}\right]\frac{iK^2}{8s}e^{2is}\left(e^{2i\psi_\beta}-1\right)$$

$$-\frac{3-\gamma}{16}\left[\frac{\gamma+1}{4(1-u_0^2)}+\frac{3-\gamma}{(1+u_0)^3}\right]\frac{i}{8s}\left(e^{2i\psi_\alpha}-1\right)$$

$$+\frac{\gamma+1}{32}\frac{\psi_\alpha-\psi_\beta}{s}\left\{-(\gamma+1)\frac{u_0}{1-u_0^2}\frac{K}{4}e^{is}e^{i(\psi_\alpha+\psi_\beta)}+\right.$$

$$\left.+\frac{1}{4}\left[\frac{3-\gamma}{4}\left(\frac{1}{1-u_0}\right)-\frac{\gamma+1}{2}\left(\frac{1}{1+u_0}\right)\right]e^{2i\psi_\alpha}+\frac{3\gamma-1}{16}\frac{e^{2is}}{1+u_0}e^{2i\psi_\beta}\right\}+$$

$$+\frac{1-u_0}{2} V_1^* \left(e^{-2i\psi_\alpha}-1\right)+\frac{1+u_0}{2} V_1^* \left(e^{-2i\psi_\beta}-1\right)+\frac{\gamma+1}{8s}\,\frac{Ae^{-2i\theta}e^{-2i\psi_\alpha}}{1-u_0}\left(\psi_\alpha-\psi_\beta\right)$$

$$-\frac{\gamma+1}{8s}\,\frac{KAe^{-2i(s+\theta)}e^{-2i\psi_\beta}}{1+u_0}\left(\psi_\alpha-\psi_\beta\right)+\frac{3-\gamma}{8s}\,\frac{iAe^{-2i\theta}}{(1+u_0)^2}\left(e^{-2i\psi_\alpha}-1\right)$$

$$+\frac{3-\gamma}{8s}\,\frac{iAKe^{-2i(\theta+s)}}{(1-u_0)^2}\left(e^{-2i\psi_\beta}-1\right)+\frac{\gamma+1}{32}(1+u_0)K_1^* e^{-2i\psi_\alpha}$$

$$+\frac{\gamma+1}{32}(1-u_0)KK_1^* e^{-is}e^{-2i\psi_\beta}-\frac{\gamma+1}{32}\left[(1+u_0)K_1^*+(1-u_0)KK_1^* e^{-is}\right]e^{-i(\psi_\alpha+\psi_\beta)}$$

$$+\frac{3-\gamma}{32}\left[\frac{1-u_0}{1+u_0}K_1^*\left(e^{-2i\psi_\alpha}-1\right)+\frac{1+u_0}{1-u_0}KK_1^* e^{-is}\left(e^{-2i\psi_\beta}-1\right)\right]+$$

$$+\frac{(\gamma+1)(3-\gamma)(3-u_0)}{8(1-u_0)^2}\,\frac{Ki}{s}\,e^{-is}\left(e^{-2i\psi_\alpha}-e^{-i(\psi_\alpha+\psi_\beta)}\right)$$

$$-\frac{(\gamma+1)(3-\gamma)(3+u_0)}{8(1+u_0)^2}\,\frac{Ki}{s}\,e^{-is}\left(e^{-2i\psi_\beta}-e^{-i(\psi_\alpha+\psi_\beta)}\right)+$$

$$+\frac{3-\gamma}{16}\left[\frac{\gamma+1}{4(1-u_0^2)}+\frac{3-\gamma}{(1-u_0)^3}\right]\frac{iK^2}{8s}\,e^{-2is}\left(e^{-2i\psi_\beta}-1\right)+$$

$$+\frac{3-\gamma}{16}\left[\frac{\gamma+1}{4(1-u_0^2)}+\frac{3-\gamma}{(1+u_0)^3}\right]\frac{i}{8s}\left(e^{-2i\psi_\alpha}-1\right)+$$

$$+\frac{\gamma+1}{32}\left(\frac{\psi_\alpha-\psi_\beta}{s}\right)\Bigg\{-(\gamma+1)\frac{u_0}{1-u_0^2}\,\frac{K}{4}\,e^{-is}e^{-i(\psi_\alpha+\psi_\beta)}+$$

$$+\frac{1}{4}\left[\frac{3-\gamma}{4}\left(\frac{1}{1-u_0}\right)-\frac{\gamma+1}{2}\left(\frac{1}{1+u_0}\right)\right]e^{-2i\psi_\alpha}+\frac{3\gamma-1}{16}\,\frac{e^{-2is}}{1+u_0}\,e^{-2i\psi_\beta}\Bigg\}$$

(50)

Three types of terms appear in this expression for t_2: periodic functions, periodic functions times the quantity $\psi_\alpha - \psi_\beta$, and terms linear in ψ_α and ψ_β. The first two types of terms are the only types that appeared in the expression for t_1 and x_1 and if properly interpreted, they indicate that if α and β (or ψ_α and ψ_β) each change by an amount equal to the period in characteristic coordinates, then the x position is unchanged and the time changes by an amount equal to the period in time coordinates. The presence of the third type of terms (linear in ψ_α and ψ_β) indicates a correction in the time period and they should not be interpreted as secular terms. This correction is a second order effect which indicates the wave travel time changes from that of a quiescent field whenever finite amplitude waves are present. A significant part of this change is attributed to the change in the mean conditions in the chamber (see Equation (48)) but there is more than just this as can be seen by setting $C_r = 0$ in Equation (50). These second order terms which are linear in ψ_α and ψ_β should be combined with the zero order result for t_0 and would indicate a change in the average slope of the characteristics due to the finite amplitude oscillations. If $\psi_\alpha - \psi_\beta$ is held constant and ψ_α and ψ_β each are changed by 2π, Equation (50) shows that the change in t_2 is the following

$$\frac{2\pi}{s}\Big\{V_2 + \frac{3-\gamma}{4}\Big[\frac{1}{(1+u_0)^2} + \frac{K}{(1-u_0)^2}\Big]C_r +$$

$$+ \frac{3-\gamma}{16}\Big(\frac{1-u_0}{1+u_0}\, is\,[K_1 - K_1^*] + \frac{1+u_0}{1-u_0}\, Kis\,[K_1 e^{-is} - K_1^* e^{is}]\Big)$$

$$+ \frac{3-\gamma}{32}\Big(\Big[\frac{\gamma+1}{4(1-u_0^2)} + \frac{3-\gamma}{(1-u_0)^3}\Big]K^2 + \Big[\frac{\gamma+1}{4(1-u_0^2)} + \frac{3-\gamma}{(1+u_0)^3}\Big]\Big)\Big\}$$

If C_p and u_o were set equal to zero and s were set equal to an integral multiple of 2π, the above term would not vanish. So it must be explained by more than change in mean conditions, nonlinear effects coupled with asymmetry of flow, or off-resonant oscillations although these all have important effects on the above term.

If the $(u_1+a_1)\frac{\partial t_1}{\partial \alpha}$ and $(u_1-a_1)\frac{\partial t_1}{\partial \beta}$ terms in Equations (6) are examined, it is seen from the results of the first order analysis that each is the product of two first harmonic terms. This is clear from Equations (39) and (41b). The products each are therefore, a second harmonic term plus a constant. After Equations (6) are integrated, the constant terms produce terms which are linear in ψ_α and ψ_β. The physical interpretation is that the wave distortion (a first order effect represented by nonzero t_1) combines with the finite amplitude (a first order effect represented by a_1 and u_1) to produce a second order change in the wave travel time. Consider a given characteristic, say a P-characteristic. At each instant its slope is affected by the value of the invariant associated with the interesting Q-characteristic since

$$\frac{d\psi}{dt} = u + a = \frac{\gamma+1}{4} P - \frac{3-\gamma}{4}(Q_o + Q')$$

where Q_o is the steady-state value of the Q invariant and Q' is the perturbation. Q_o has the same value for all Q characteristics. P is a constant for the given characteristic so Q' is the critical term which determines the variation in slope. Q' is a trigonometric function in characteristic coordinates so that its integral over a period is zero. Due to wave distortion, however, this is not so in space vs. time coordinates and this integral of Q' may be nonzero meaning that the integral of the slope, or in other words, the average slope, may be different from its steady-state value.

THIRD ORDER SOLUTION

It is necessary to perform a third order analysis since the first order coefficients of the series (42) were found to be zero from the second order analysis. The perturbations of the parameters ϕ (or τ), n, and s (or ω) are therefore of second order and the second order coefficients are determined by a third order analysis.

If the series (42) is substituted in Equation (14) (with $n^{(1)} = \phi^{(1)} = s^{(1)} = 0$), if the transformation indicated by Equation (35) is performed, and if the third order terms are separated, the following equation results

$$K\left[1+u_0\left(1-\gamma_n^{(0)}\right)\right]Q_3\left(s^{(0)}+\psi/\beta\right)-\left[1-u_0\left(1-\gamma n^{(0)}\right)\right]Q_3\left(\psi/\beta\right)+$$

$$+\gamma n^{(0)}u_0\left[KQ_3\left(s^{(0)}+\psi/\beta-\phi^{(0)}\right)+Q_3\left(\psi/\beta-\phi^{(0)}\right)\right]=2u_0 R_3 \tag{51}$$

where the definition is made

$$R_3 \equiv R_3' - \left\{-KQ_1\left(s^{(0)}+\psi/\beta\right)-Q_1\left(\psi/\beta\right)+KQ_1\left(s^{(0)}+\psi/\beta-\phi^{(0)}\right)+Q_1\left(\psi/\beta-\phi^{(0)}\right)\right\}\frac{\gamma n^{(2)}}{2}$$

$$-K\left\{\left[1+u_0\left(1-\gamma n^{(0)}\right)\right]\frac{dQ_1}{d\psi/\beta}\left(s^{(0)}+\psi/\beta\right)+\gamma n^{(0)}u_0\frac{dQ_1}{d\psi/\beta}\left(s^{(0)}+\psi/\beta-\phi^{(0)}\right)\right\}\frac{s^{(2)}}{2u_0}$$

$$+\left\{K\frac{dQ_1}{d\psi/\beta}\left(s^{(0)}+\psi/\beta-\phi^{(0)}\right)+\frac{dQ_1}{d\psi/\beta}\left(\psi/\beta-\phi^{(0)}\right)\right\}\frac{\gamma n^{(0)}}{2}\phi^{(2)}$$

The R_3' term is given by Equation (E-17) and may be evaluated by substitution for a_1, u_1, u_2, a_2, t_1 and t_2 from Equations (39), (48), (41), and (50). These quantities are evaluated at the combustion zone; i.e., along the line $\psi_\alpha = \psi_\beta$ (or $\alpha = \beta$). They are substituted in the exponential form for the purpose of simplifying the associated operations. The R_3' term contains many terms which will be evaluated one by one before Equation (51) is solved for ρ_3.

The first seventeen terms may be evaluated by use of Equations (39) and (48). The first term* is $2\left[\left(\frac{2\gamma n}{\gamma-1}\right)^2 - M\right] a_{2_{b_0}} a_{1_{b_0}}$. a_2 and a_1 are evaluated at $\psi_\beta - \phi^{(0)}, \psi_\beta - \phi^{(0)}$ (or at $\beta - b_0^{(0)}, \beta - b_0^{(0)}$). The error in using $\phi^{(0)}$ rather than ϕ is of second order and would appear only as a fifth order effect in the analysis. The final result for this term is

$$2\left[\left(\frac{2\gamma n}{\gamma-1}\right)^2 - M\right] a_{2_{b_0}} a_{1_{b_0}} = m_1 \Big\{ (c_1 + i d_1) e^{3i\psi_\beta} + (c_1 - i d_1) e^{-3i\psi_\beta} + (g_1 + i h_1) e^{i\psi_\beta} + (g_1 - i h_1) e^{-i\psi_\beta} \Big\}$$

where

$$m_1 \equiv \left[\frac{\gamma^2 n^2}{4} - M\left(\frac{\gamma-1}{4}\right)^2\right]$$

$$c_1 \equiv A\left[\cos(2\theta - 3\phi) + K\cos(2\theta + 2s - 3\phi) + K\cos(2\theta + s - 3\phi) + K^2\cos(2\theta + 3s - 3\phi)\right]$$

$$d_1 \equiv A\left[\sin(2\theta - 3\phi) + K\sin(2\theta + 2s - 3\phi) + K\sin(2\theta + s - 3\phi) + K^2\sin(2\theta + 3s - 3\phi)\right]$$

$$g_1 \equiv A\left[\cos(2\theta - \phi) + K\cos(2\theta + 2s - \phi) + K\cos(2\theta - \phi - s)\right.$$

*In accordance with the short-hand notation of Appendix E, a_1 represents $a_1(\psi_\beta, \psi_\beta)$ and $a_{1_{b_0}}$ represents $a_1(\psi_\beta - \phi, \psi_\beta - \phi)$.

$$+ K^2 \cos (2\Theta + s - \phi)] + (1+K) C_r [\cos \phi + K \cos (s-\phi)]$$

$$h_1 \equiv A[\sin (2\theta - \phi) + K \sin (2\theta + 2s - \phi) + K \sin (2\theta - \phi - s)$$

$$+ K^2 \sin (2\theta + s - \phi)] + (1+K) C_r [K \sin (s-\phi) - \sin \phi]$$

It is understood in the above relationships ϕ is $\phi^{(0)}$, s is $s^{(0)}$, and n is $n^{(0)}$. This is true for the expressions to be developed for the other terms of R_3^1.

The second term of R_3^1 is also found by substitution from Equations (39) for $a_1(\psi/\beta, \psi/\beta)$ and (48) for $a_2(\psi/\beta - \phi, \psi/\beta - \phi)$ with the result

$$-\left(\frac{2\sigma n}{\gamma-1}\right)^2 a_{2_{b_0}} a_1 = m_2 \left\{ (c_2 + i d_2) e^{3i\psi/\beta} + \right.$$

$$\left. + (c_2 - i d_2) e^{-3i\psi/\beta} + (g_2 + i h_2) e^{i\psi/\beta} + (g_2 - i h_2) e^{-i\psi/\beta} \right\}$$

$$m_2 \equiv -\frac{\gamma^2 n^2}{8}$$

$$c_2 \equiv A[\cos (2\theta - 2\phi) + K \cos (2\theta + 2s - 2\phi) +$$

$$+ K \cos (2\theta - 2\phi + s) + K^2 \cos (2\theta - 2\phi + 3s)]$$

$$d_2 \equiv A[\sin (2\theta - 2\phi) + K \sin (2\theta + 2s - 2\phi) +$$

$$+ K \sin (2\theta - 2\phi + s) + K^2 \sin (2\theta - 2\phi + 3s)]$$

$$g_2 \equiv A[\cos (2\theta - 2\phi) + K \cos (2\theta + 2s - 2\phi) +$$

$$+ K \cos (2\theta - 2\phi - s) + K^2 \cos (2\theta - 2\phi + s)] + (1+K) C_r (1 + K \cos s)$$

$$h_2 \equiv A\left[\sin(2\theta-2\phi)+K\sin(2\theta+2s-2\phi)+\right.$$
$$\left.+K\sin(2\theta-2\phi-s)+K^2\sin(2\theta-2\phi+s)\right]+(1+K)C_r K\sin s$$

The next fifteen terms are developed by substitution in similar fashion. The results of these substitutions are

Third Term:

$$-\left(\frac{2\gamma n}{\gamma-1}\right)^2 a_2\, a_{1_{b_0}} = m_3\left\{(c_3+id_3)e^{3i\psi_\beta}+(c_3-id_3)e^{-3i\psi_\beta}\right.$$
$$\left.+(g_3+ih_3)e^{i\psi_\beta}+(g_3-ih_3)e^{-i\psi_\beta}\right\}$$

where

$$m_3 \equiv -\frac{\gamma^2 n^2}{8}$$

$$c_3 \equiv A\left[\cos(2\theta-\phi)+K\cos(2\theta+2s-\phi)+\right.$$
$$\left.+K\cos(2\theta+s-\phi)+K^2\cos(2\theta+3s-\phi)\right]$$

$$d_3 \equiv A\left[\sin(2\theta-\phi)+K\sin(2\theta+2s-\phi)\right.$$
$$\left.+K\sin(2\theta+s-\phi)+K^2\sin(2\theta+3s-\phi)\right]$$

$$g_3 \equiv A\left[\cos(2\theta+\phi)+K\cos(2\theta+2s+\phi)+\right.$$
$$\left.+K\cos(2\theta-s+\phi)+K^2\cos(2\theta+s+\phi)\right]+$$
$$+(1+K)C_r\left[\cos\phi+K\cos(s-\phi)\right]$$

$$h_3 \equiv A\left[\sin(2\theta+\phi)+K\sin(2\theta+2s+\phi)+\right.$$
$$\left.+K\sin(2\theta-s+\phi)+K^2\sin(2\theta+s+\phi)\right]+$$
$$+(1+K)C_r\left[K\sin(s-\phi)-\sin\phi\right]$$

Fourth Term:

$$2\left[M-\frac{3-\gamma}{(\gamma-1)^2}\right] a_2\, a_1 = m_4 \Big\{ (c_4 + i d_4) e^{3i\psi_\beta} + (c_4 - i d_4) e^{-3i\psi_\beta}$$

$$+ (g_4 + i h_4) e^{i\psi_\beta} + (g_4 - i h_4) e^{-i\psi_\beta} \Big\}$$

where

$$m_4 \equiv \left[M \left(\frac{\gamma-1}{4}\right)^2 - \frac{3-\gamma}{16}\right]$$

$$c_4 \equiv A\Big[\cos 2\theta + K \cos(2\theta + 2s) + K \cos(2\theta + s)$$

$$+ K^2 \cos(2\theta + 3s)\Big]$$

$$d_4 \equiv A\Big[\sin 2\theta + K \sin(2\theta + 2s) + K \sin(2\theta + s)$$

$$+ K^2 \sin(2\theta + 3s)\Big]$$

$$g_4 \equiv A\Big[\cos 2\theta + K \cos(2\theta + 2s) + K \cos(2\theta - s)$$

$$+ K^2 \cos(2\theta + s)\Big] + (1+K)\, C_r (1 + K \cos s)$$

$$h_4 \equiv A\Big[\sin 2\theta + K \sin(2\theta + 2s) + K \sin(2\theta - s)$$

$$+ K^2 \sin(2\theta + s)\Big] + (1+K)\, C_r\, K \sin s$$

Fifth Term:

$$\left[\frac{4\gamma n}{\gamma-1} M - \left(\frac{2\gamma n}{\gamma-1}\right)^3 - L\right] a_{1_{b_0}}^{3} = m_5 \Big\{ (c_5 + i d_5) e^{3i\psi_\beta} +$$

$$+ (c_5 - i d_5) e^{-3i\psi_\beta} + (g_5 + i h_5) e^{i\psi_\beta} + (g_5 - i h_5) e^{-i\psi_\beta} \Big\}$$

where

$$m_5 \equiv \left[\frac{\gamma n}{2} M \left(\frac{\gamma-1}{8}\right)^2 - \left(\frac{\gamma n}{4}\right)^3 - L \left(\frac{\gamma-1}{8}\right)^3\right]$$

$$c_5 \equiv [\cos 3\phi + 3K \cos(s-3\phi) + 3K^2 \cos(2s-3\phi)$$
$$+ K^3 \cos(3s-3\phi)]$$

$$d_5 \equiv [-\sin 3\phi + 3K \sin(s-3\phi) + 3K^2 \sin(2s-3\phi)$$
$$+ K^3 \sin(3s-3\phi)]$$

$$g_5 \equiv 3[(2K^2+1)\cos\phi + (2K+K^3)\cos(s-\phi) +$$
$$+ K^2 \cos(2s-\phi) + K\cos(s+\phi)]$$

$$h_5 \equiv 3[-(2K^2+1)\sin\phi + (2K+K^3)\sin(s-\phi) +$$
$$+ K^2 \sin(2s-\phi) - K\sin(s+\phi)]$$

Sixth Term:

$$\left[\left(\frac{2\gamma n}{\gamma-1}\right)^3 - \frac{2\gamma n M}{\gamma-1}\right]\left(a_{1_{b_0}}\right)^2 a_1 = m_6 \left\{(c_6 + i d_6)e^{3i\psi_\beta} + \right.$$
$$\left. + (c_6 - i d_6)e^{-3i\psi_\beta} + (g_6 + i h_6)e^{i\psi_\beta} + (g_6 - i h_6)e^{-i\psi_\beta}\right\}$$

where

$$m_6 \equiv \left[\left(\frac{\gamma n}{4}\right)^3 - \frac{\gamma n}{4} M \left(\frac{\gamma-1}{8}\right)^2\right]$$

$$c_6 \equiv [\cos 2\phi + 3K\cos(s-2\phi) + 3K^2\cos(2s-2\phi) +$$
$$+ K^3 \cos(3s-2\phi)]$$

$$d_6 \equiv [-\sin 2\phi + 3K\sin(s-2\phi) + 3K^2\sin(2s-2\phi) +$$
$$+ K^3\sin(3s-2\phi)]$$

$$g_6 \equiv \Big[(2K^2+1)\cos 2\phi + (2K+K^3)\cos(s-2\phi) +$$

$$+ K^2 \cos(2s-2\phi) + K\cos(s+2\phi) + 2(2K^2+1) +$$

$$+ 2(2K+K^3)\cos s + 2K^2\cos 2s + 2K\cos s\Big]$$

$$h_6 \equiv \Big[-(2K^2+1)\sin 2\phi + (2K+K^3)\sin(s-2\phi) +$$

$$+ K^2\sin(2s-2\phi) - K\sin(s+2\phi) + 2(2K+K^3)\sin s$$

$$+ 2K^2\sin 2s - 2K\sin s\Big]$$

Seventh Term:

$$-\frac{2\gamma n M}{\gamma-1}\, a_1^{\,2} a_{1_6} = m_7\Big\{(c_7+id_7)e^{3i\psi_\beta} + (c_7-id_7)e^{-3i\psi_\beta}$$

$$+ (g_7+ih_7)e^{i\psi_\beta} + (g_7-ih_7)e^{-i\psi_\beta}\Big\}$$

where

$$m_7 \equiv -\frac{\gamma n M}{4}\left(\frac{\gamma-1}{\gamma}\right)^2$$

$$c_7 \equiv \Big[\cos\phi + 3K\cos(s-\phi) + 3K^2\cos(2s-\phi) + K^3\cos(3s-\phi)\Big]$$

$$d_7 \equiv \Big[-\sin\phi + 3K\sin(s-\phi) + 3K^2\sin(2s-\phi) + K^3\sin(3s-\phi)\Big]$$

$$g_7 \equiv \Big[3(2K^2+1)\cos\phi + (2K+K^3)\Big(\cos(s+\phi) + 2\cos(s-\phi)\Big)$$

$$+ K^2\Big(\cos(2s+\phi) + 2\cos(2s-\phi)\Big) +$$

$$+ K\Big(\cos(s-\phi) + 2\cos(s+\phi)\Big)\Big]$$

$$h_7 \equiv \Big[-(2K^2+1)\sin\phi + (2K+K^3)\Big(\sin(s+\phi) + 2\sin(s-\phi)\Big)$$

$$+K^2\Big(\sin(2s+\phi) + 2\sin(2s-\phi)\Big) - K\Big(\sin(s-\phi) +$$

$$+2\sin(s+\phi)\Big)\Big]$$

Eighth Term:

$$\left[L - \frac{2}{3}\ \frac{(3-\gamma)(2-\gamma)}{(\gamma-1)^3}\right] a_1^{\,3} = m_8 \Big\{(c_8 + i d_8) e^{3i\psi_\beta} +$$

$$+ (c_8 - i d_8) e^{-3i\psi_\beta} + (g_8 + i h_8) e^{i\psi_\beta} + (g_8 - i h_8) e^{-i\psi_\beta}\Big\}$$

where

$$m_8 \equiv \left[L\left(\frac{\gamma-1}{8}\right)^3 - \frac{2}{3}\ \frac{(3-\gamma)(2-\gamma)}{(8)^3}\right]$$

$$c_8 \equiv \left[1 + 3K\cos s + 3K^2\cos 2s + K^3\cos 3s\right]$$

$$d_8 \equiv \left[3K\sin s + 3K^2\sin 2s + K^3\sin 3s\right]$$

$$g_8 \equiv 3\left[(2K^2+1) + (3K+K^3)\cos s + K^2\cos 2s\right]$$

$$h_8 \equiv 3\left[(K+K^3)\sin s + K^2\sin 2s\right]$$

Ninth Term:

$$-\frac{2}{\gamma-1}\ a_2\ \frac{u_1}{u_0} = m_9 \Big\{(c_9 + i d_9) e^{3i\psi_\beta} + (c_9 - i d_9) e^{-3i\psi_\beta}$$

$$+ (g_9 + i h_9) e^{i\psi_\beta} + (g_9 - i h_9) e^{-i\psi_\beta}\Big\}$$

where

$$m_9 \equiv -\frac{1}{8u_0}$$

$$c_9 \equiv A\left[K \cos(2\theta+s) + K^2 \cos(2\theta+3s) - \cos 2\theta \right.$$
$$\left. - K \cos 2(\theta+s)\right]$$

$$d_9 \equiv A\left[K \sin(2\theta+s) + K^2 \sin(2\theta+3s) - \sin 2\theta \right.$$
$$\left. - K \sin 2(\theta+s)\right]$$

$$g_9 \equiv A\left[K \cos(2\theta-s) + K^2 \cos(2\theta+s) - \cos 2\theta - \right.$$
$$\left. - K \cos 2(\theta+s)\right] + (1+K)C_r(K \cos s - 1)$$

$$h_9 \equiv A\left[K \sin(2\theta-s) + K^2 \sin(2\theta+s) - \sin 2\theta - \right.$$
$$\left. - K \sin 2(\theta+s)\right] + (1+K)C_r \, K \sin s$$

Tenth Term:

$$-\frac{3-\gamma}{(\gamma-1)^2} a_1^2 \frac{u_1}{u_0} = m_{10} \left\{ (c_{10} + i d_{10}) e^{3i\psi_p} + \right.$$
$$\left. (c_{10} - i d_{10}) e^{-3i\psi_p} + (g_{10} + i h_{10}) e^{i\psi_p} + (g_{10} - i h_{10}) e^{-i\psi_p} \right\}$$

where

$$m_{10} \equiv -\frac{3-\gamma}{256 u_0}$$

$$c_{10} \equiv \left[-K \cos s - 1 + K^2 \cos 2s + K^3 \cos 3s\right]$$

$$d_{10} \equiv \left[-K \sin s + K^2 \sin 2s + K^3 \sin 3s\right]$$

$$g_{10} \equiv \left[(2K^2 - 3) + 3(K^3 - K)\cos s + K^2 \cos 2s\right]$$

$$h_{10} \equiv \left[(-K + 3K^3)\sin s + K^2 \sin 2s\right]$$

Eleventh Term:

$$-\frac{2}{\gamma-1}\, a_1 \frac{u_2}{u_0} = m_{11}\left\{(c_{11}+id_{11})e^{3i\psi_\beta} + \right.$$

$$\left. + (c_{11}-id_{11})e^{-3i\psi_\beta} + (g_{11}+ih_{11})e^{i\psi_\beta} + (g_{11}-ih_{11})e^{-i\psi_\beta}\right\}$$

where

$$m_{11} = -\frac{1}{8u_0}$$

$$c_{11} \equiv A\Big[K\cos(2\theta+2s) + K^2\cos(2\theta+3s) - \cos 2\theta$$

$$-K\cos(2\theta+s)\Big]$$

$$d_{11} \equiv A\Big[K\sin(2\theta+2s) + K^2\sin(2\theta+3s) - \sin 2\theta$$

$$-K\sin(2\theta+s)\Big]$$

$$g_{11} \equiv A\Big[K\cos(2\theta+2s) + K^2\cos(2\theta+s) -$$

$$-\cos 2\theta - K\cos(2\theta-s)\Big] + (K-1)C_r(1+K\cos s)$$

$$h_{11} \equiv A\Big[K\sin(2\theta+2s) + K^2\sin(2\theta+s) -$$

$$-\sin 2\theta - K\sin(2\theta-s)\Big] + (K-1)C_r\, K\sin s$$

Twelfth Term:

$$-\left(\frac{2\gamma n}{\gamma-1}\right)^2 \frac{da_2}{d\psi_\beta}\Big|_{b_0}\int_{\psi_\beta-\phi}^{\psi_\beta} a_1\, d\psi_\beta' = m_{12}\left\{(c_{12}+id_{12})e^{3i\psi_\beta}\right.$$

$$\left. + (c_{12}-id_{12})e^{-3i\psi_\beta} + (g_{12}+ih_{12})e^{i\psi_\beta} + (g_{12}-ih_{12})e^{-i\psi_\beta}\right\}$$

where

$$m_{12} \equiv -\left(\frac{\gamma n}{2}\right)^2$$

$$c_{12} \equiv A\Big[\cos 2(\theta-\phi)+K\cos 2(\theta+s-\phi)+K\cos(2\theta-2\phi+s)$$

$$+K^2\cos(2\theta-2\phi+3s)-\cos(2\theta-3\phi)-K\cos(2\theta+2s-3\phi)$$

$$-K\cos(2\theta-3\phi+s)-K^2\cos(2\theta+3s-3\phi)\Big]$$

$$d_{12} \equiv A\Big[\sin 2(\theta-\phi)+K\sin 2(\theta+s-\phi)+K\sin(2\theta-2\phi+s)$$

$$+K^2\sin(2\theta-2\phi+3s)-\sin(2\theta-3\phi)-K\sin(2\theta+2s-3\phi)$$

$$-K\sin(2\theta-3\phi+s)-K^2\sin(2\theta+3s-3\phi)\Big]$$

$$g_{12} \equiv A\Big[\cos(2\theta-\phi)+K\cos(2\theta+2s-\phi)+$$

$$+K\cos(2\theta-s-\phi)+K^2\cos(2\theta+s-\phi)-\cos(2\theta-2\phi)$$

$$-K\cos(2\theta+2s-2\phi)-K\cos(2\theta-2\phi-s)-K^2\cos(2\theta-2\phi+s)\Big]$$

$$h_{12} \equiv A\Big[\sin(2\theta-\phi)+K\sin(2\theta+2s-\phi)+K\sin(2\theta-s-\phi)$$

$$+K^2\sin(2\theta+s-\phi)-\sin(2\theta-2\phi)-K\sin(2\theta+2s-2\phi)$$

$$-K\sin(2\theta-2\phi-s)-K^2\sin(2\theta-2\phi+s)\Big]$$

Thirteenth Term:

$$\left[3\left(\frac{2\gamma n}{\gamma-1}\right)^3-\frac{4\gamma n M}{\gamma-1}\right]a_{1_{b_0}}\frac{da_1}{d\psi_\beta}{}_{b_0}\int_{\psi_\beta-\phi}^{\psi_\beta} a_1\, d\psi_\beta' =$$

$$m_{13}\left\{(c_{13}+id_{13})e^{3i\psi_\beta}+(c_{13}-id_{13})e^{-3i\psi_\beta}+(g_{13}+ih_{13})e^{i\psi_\beta}+(g_{13}-ih_{13})e^{-i\psi_\beta}\right\}$$

where

$$m_{13}=\left[3\left(\frac{\gamma n}{4}\right)^3-\frac{\gamma n(\gamma-1)^2}{128}M\right]$$

$$c_{13} \equiv \Big[\cos 2\phi + 3K \cos(s-2\phi) + 3K^2 \cos(2s-2\phi)$$

$$+ K^3 \cos(3s-2\phi) - \cos 3\phi - 3K\cos(s-3\phi) -$$

$$- 3K^2 \cos(2s-3\phi) - K^3 \cos(3s-3\phi)\Big]$$

$$d_{13} \equiv \Big[-\sin 2\phi + 3K \sin(s-2\phi) + 3K^2 \sin(2s-2\phi)$$

$$+ K^3 \sin(3s-2\phi) + \sin 3\phi - 3K \sin(s-3\phi)$$

$$- 3K^2 \sin(2s-3\phi) - K^3 \sin(3s-3\phi)\Big]$$

$$g_{13} \equiv \Big[(1+2K^2)\cos\phi + (2K+K^3)\cos(s-\phi) +$$

$$+ K^2\cos(2s-\phi) + K\cos(s+\phi) - (1+2K^2)\cos 2\phi$$

$$-(2K+K^3)\cos(s-2\phi) - K^2\cos(2s-2\phi) - K\cos(s+2\phi)\Big]$$

$$h_{13} \equiv \Big[-(1+2K^2)\sin\phi + (2K+K^3)\sin(s-\phi) +$$

$$+ K^2\sin(2s-\phi) - K\sin(s+\phi) + (1+2K^2)\sin 2\phi -$$

$$-(2K+K^3)\sin(s-2\phi) - K^2\sin(2s-2\phi) + K\sin(s+2\phi)\Big]$$

Fourteenth Term:

$$-\left(\frac{2\gamma n}{\gamma-1}\right)^2 \frac{da_1}{d\psi_\beta}\Big|_{b_0} \int_{\psi_\beta-\phi}^{\psi_\beta} a_2\, d\psi'_\beta = m_{14}\Big\{(c_{14}+id_{14})e^{3i\psi_\beta} +$$

$$+ (c_{14}-id_{14})e^{-3i\psi_\beta} + (g_{14}+ih_{14})e^{i\psi_\beta} + (g_{14}-ih_{14})e^{-i\psi_\beta}\Big\}$$

where

$$m_{14} = -\left(\frac{\gamma n}{4}\right)^2$$

$$c_{14} \equiv A\Big[\cos(2\theta-\phi) + K\cos(2\theta+2s-\phi) +$$

$$+ K\cos(2\theta+s-\phi) + K^2\cos(2\theta+3s-\phi) - \cos(2\theta-3\phi)$$

$$-K\cos(2\theta+2s-3\phi) - K\cos(2\theta+s-3\phi) - K^2\cos(2\theta+3s-3\phi)\Big]$$

$$d_{14} \equiv A\Big[\sin(2\theta-\phi) + K\sin(2\theta+2s-\phi) +$$

$$+ K\sin(2\theta+s-\phi) + K^2\sin(2\theta+3s-\phi) - \sin(2\theta-3\phi)$$

$$-K\sin(2\theta+2s-3\phi) - K\sin(2\theta+s-3\phi) - K^2\sin(2\theta+3s-3\phi)\Big]$$

$$g_{14} \equiv A\Big[\cos(2\theta-\phi) + K\cos(2\theta+2s-\phi) + K\cos(2\theta-s-\phi)$$

$$+ K^2\cos(2\theta+s-\phi) - \cos(2\theta+\phi) - K\cos(2\theta+2s+\phi)$$

$$-K\cos(2\theta-s+\phi) - K^2\cos(2\theta+s+\phi)\Big] +$$

$$+ 2(1+K)C_r\,\phi\Big[\sin\phi - K\sin(s-\phi)\Big]$$

$$h_{14} \equiv A\Big[\sin(2\theta-\phi) + K\sin(2\theta+2s-\phi) + K\sin(2\theta-s-\phi)$$

$$+ K^2\sin(2\theta+s-\phi) - \sin(2\theta+\phi) - K\sin(2\theta+2s+\phi)$$

$$-K\sin(2\theta-s+\phi) - K^2\sin(2\theta+s+\phi)\Big]$$

$$+ 2(1+K)C_r\,\phi\Big[\cos\phi + K\cos(s-\phi)\Big]$$

Fifteenth Term:

$$-\left(\frac{2\gamma n M}{\gamma-1}\right)\frac{da_1}{d\psi_\beta}b_0\int_{\psi_\beta-\phi}^{\psi_\beta} a_1^2\,d\psi_\beta' = m_{15}\Big\{(c_{15}+id_{15})e^{3i\psi_\beta}+$$

$$+(c_{15}-id_{15})e^{-3i\psi_\beta}+(g_{15}+ih_{15})e^{i\psi_\beta}+(g_{15}-ih_{15})e^{-i\psi_\beta}\Big\}$$

where

$$m_{15}\equiv-\frac{\gamma n M}{4}\left(\frac{\gamma-1}{8}\right)^2$$

$$c_{15}\equiv\frac{1}{2}\Big[\cos\phi+3K\cos(s-\phi)+3K^2\cos(2s-\phi)-\cos 3\phi$$

$$-3K\cos(s-3\phi)+K^3\cos(3s-\phi)-K^3\cos(3s-3\phi)-3K^2\cos(2s-3\phi)\Big]$$

$$d_{15}\equiv\frac{1}{2}\Big[-\sin\phi+3K\sin(s-\phi)+3K^2\sin(2s-\phi)+\sin 3\phi-$$

$$-3K\sin(s-3\phi)+K^3\sin(3s-\phi)-K^3\sin(3s-3\phi)-3K^2\sin(2s-3\phi)\Big]$$

$$g_{15}\equiv\left[1+2K\cos s+K^2\right]K\sin\phi\sin s+$$

$$+2\phi\Big[(1+2K^2)\sin\phi+K\sin(s+\phi)-(2K+K^3)\sin(s-\phi)$$

$$-K^2\sin(2s-\phi)\Big]$$

$$h_{15}\equiv-\left[1+2K\cos s+K^2\right](\sin\phi)(1+K\cos s)+$$

$$+2\phi\Big[(1+2K^2)\cos\phi+(2K+K^3)\cos(s-\phi)+K^2\cos(2s-\phi)$$

$$+K\cos(s+\phi)\Big]$$

Sixteenth Term:

$$-\frac{1}{2}\left(\frac{2\gamma n}{\gamma-1}\right)^3 \frac{d^2 a_1}{d\psi_\beta^2} b_0 \left[\int_{\psi_\beta-\phi}^{\psi_\beta} a_1\, d\psi_\beta\right]^2 = m_{16}\Big\{(c_{16}+id_{16})e^{3i\psi_\beta}$$

$$+(c_{16}-id_{16})e^{-3i\psi_\beta} + (g_{16}+ih_{16})e^{i\psi_\beta} + (g_{16}-ih_{16})e^{-i\psi_\beta}\Big\}$$

where

$$m_{16} \equiv -\frac{1}{2}\left(\frac{\gamma n}{4}\right)^3$$

$$c_{16} \equiv \left[\cos\phi + 3K\cos(s-\phi) + 3K^2\cos(2s-\phi) + K^3\cos(3s-\phi)\right]$$

$$-2\left[\cos 2\phi + 3K\cos(s-2\phi) + 3K^2\cos(2s-2\phi) + K^3\cos(3s-2\phi)\right]$$

$$+\left[\cos 3\phi + 3K\cos(s-3\phi) + 3K^2\cos(2s-3\phi) + K^3\cos(3s-3\phi)\right]$$

$$d_{16} \equiv \left[-\sin\phi + 3K\sin(s-\phi) + 3K^2\sin(2s-\phi) + K^3\sin(3s-\phi)\right]$$

$$-2\left[-\sin 2\phi + 3K\sin(s-2\phi) + 3K^2\sin(2s-2\phi) + K^3\sin(3s-2\phi)\right]$$

$$+\left[-\sin 3\phi + 3K\sin(s-3\phi) + 3K^2\sin(2s-3\phi) + K^3\sin(3s-3\phi)\right]$$

$$g_{16} \equiv \left[2(2K^2+1)(\cos 2\phi - \cos\phi) + (K^3-K)\cos(s+\phi) -\right.$$

$$\left.-(5K+3K^3)\cos(s-\phi) + 2(2K+K^3)\cos(s-2\phi) + 2K\cos(s+2\phi)\right]$$

$$+K^2\left[\cos(2s+\phi) - 3\cos(2s-\phi) + 2\cos 2(s-\phi)\right]$$

$$h_{16} \equiv \left[2(2K^2+1)(2\sin\phi - \sin 2\phi) + (5K+K^3)\sin(s+\phi)\right.$$

$$-(7K+3K^3)\sin(s-\phi)+2(2K+K^3)\sin(s-2\phi)-$$

$$-2K\sin(s-2\phi)+K^2\left[\sin(2s+\phi)-3\sin(2s-\phi)+2\sin 2(s-\phi)\right]$$

Seventeenth Term:

$$-\left(\frac{2\gamma n}{\gamma-1}\right)^3 a_1 \frac{da_1}{d\psi_\beta}\Big|_0 \int_{\psi_\beta-\phi}^{\psi_\beta} a_1\, d\psi'_\beta = m_{17}\Big\{(c_{17}+id_{17})e^{3i\psi_\beta}+$$

$$+(c_{17}-id_{17})e^{-3i\psi_\beta}+(g_{17}+ih_{17})e^{i\psi_\beta}+(g_{17}-ih_{17})e^{-i\psi_\beta}\Big\}$$

where

$$m_{17} \equiv -\left(\frac{\gamma n}{4}\right)^3$$

$$c_{17} \equiv \cos\phi+3K\cos(s-\phi)+3K^2\cos(2s-\phi)+K^3\cos(3s-\phi)$$

$$-\cos 2\phi-3K\cos(s-2\phi)-3K^2\cos 2(s-\phi)-K^3\cos(3s-2\phi)$$

$$d_{17} \equiv -\sin\phi+3K\sin(s-\phi)+3K^2\sin(2s-\phi)+K^3\sin(3s-\phi)$$

$$+\sin 2\phi-3K\sin(s-2\phi)-3K^2\sin 2(s-\phi)-K^3\sin(3s-2\phi)$$

$$g_{17} \equiv (2K^2+1)(2-\cos\phi-\cos 2\phi)+(2K+K^3)\Big(2\cos s-$$

$$-\cos(s+\phi)-\cos(s-2\phi)\Big)+K^2\Big(2\cos 2s-\cos(2s+\phi)-$$

$$-\cos 2(s-\phi)\Big)+K\Big(2\cos s-\cos(s-\phi)-\cos(s+2\phi)\Big)$$

$$h_{17} \equiv (2K^2+1)(\sin 2\phi-\sin\phi)+(2K+K^3)\Big(2\sin s-$$

$$-\sin(s+\phi)-\sin(s-2\phi)\Big)+K^2\Big(2\sin 2s-\sin(2s+\phi)$$

$$-\sin 2(s-\phi)\Big)+K\Big(-2\sin s+\sin(s-\phi)+\sin(s+2\phi)\Big)$$

The rest of the terms appearing in R_3' result from coordinate transformation (see Appendix E) and involve t_1, t_2 and $\frac{dt_1}{d\psi_\beta}$ evaluated at the points ψ_β, ψ_β and $\psi_\beta - \phi, \psi_\beta - \phi$. The following is the first of these terms:

$$\Big\{\frac{\gamma n}{\gamma-1} s^2(1-u_0^2)\frac{dt_1}{d\psi_\beta}{}_{b_0}\frac{da_1}{d\psi_\beta}{}_{b_0} - \frac{2\gamma n}{\gamma-1} s\frac{da_2}{d\psi_\beta}{}_{b_0} - \left(\frac{2\gamma n}{\gamma-1}\right)^2 s\frac{d^2a_1}{d\psi_\beta^2}{}_{b_0}\int_{\psi_\beta-\phi}^{\psi_\beta} a_1 d\psi_\beta$$

$$- \frac{\gamma n}{\gamma-1}\left(\frac{1-u_0}{2}\right)s^2\frac{d^2a_1}{d\psi_\beta^2}{}_{b_0}\left[t_1 - t_{1_{b_0}}\right] + \left[3\left(\frac{2\gamma n}{\gamma-1}\right)^2 - 2M\right]a_{1_{b_0}}\, s\frac{da_1}{d\psi_\beta}{}_{b_0}$$

$$+ \left[\frac{4\gamma n}{(\gamma-1)^2} - \left(\frac{2\gamma n}{\gamma-1}\right)^2\right]a_1\, s\frac{da_1}{d\psi_\beta}{}_{b_0}\Big\}\left(\frac{1-u_0^2}{2}\right)\left[t_1 - t_{1_{b_0}}\right]$$

Equations (39), (41), and (48) are used to substitute in the above relationship. As a preliminary step in evaluating the above term, we find

$$\Big\{\frac{\gamma n}{\gamma-1} s^2(1-u_0^2)\frac{dt_1}{d\psi_\beta}{}_{b_0}\frac{da_1}{d\psi_\beta}{}_{b_0} - \frac{2\gamma n}{\gamma-1} s\frac{da_2}{d\psi_\beta}{}_{b_0} - \left(\frac{2\gamma n}{\gamma-1}\right)^2 s\frac{d^2a_1}{d\psi_\beta^2}{}_{b_0}\int_{\psi_\beta-\phi}^{\psi_\beta} a_1 d\psi_\beta$$

$$- \frac{\gamma n}{\gamma-1}\left(\frac{1-u_0}{2}\right)s^2\frac{d^2a_1}{d\psi_\beta^2}{}_{b_0}\left[t_1 - t_{1_{b_0}}\right] + \left[3\left(\frac{2\gamma n}{\gamma-1}\right)^2 - 2M\right]s\, a_{1_{b_0}}\frac{da_1}{d\psi_\beta}{}_{b_0} +$$

$$+ \left[\frac{4\gamma n}{(\gamma-1)^2} - \left(\frac{2\gamma n}{\gamma-1}\right)^2\right]s a_1\frac{da_1}{d\psi_\beta}{}_{b_0}\Big\} = (z_1 + iz_2)e^{2i\psi_\beta} +$$

$$+ (z_1 - iz_2)e^{-2i\psi_\beta} + z_3$$

where the definitions are made as follows:

$$z_1 \equiv -\gamma n(1-u_0^2)\Big\{\frac{(\gamma+1)s^2}{128}\frac{\left[K\left(\frac{1}{1+u_0}\right)^2 + \left(\frac{1}{1-u_0}\right)^2\right]}{1-\cos s}\left[(1-K)\cos(s-2\phi) + K\cos 2(s-\phi)\right)$$

$$- \cos 2\phi\Big] + \frac{(3-\gamma)s}{64}\Big[\frac{K}{(1-u_0)^2}\Big(\sin(s-2\phi) + K\sin 2(s-\phi)\Big) +$$

$$+ \frac{K\sin(s-2\phi) - \sin 2\phi}{(1+u_0)^2}\Big]\Big\} + \gamma n\, A s\Big[\sin 2(\theta-\phi) + K\sin 2(\theta+s-\phi)\Big]$$

$$+ \left(\frac{\gamma n}{4}\right)^2 s\Big[-\sin\phi + 2K\sin(s-\phi) + K^2\sin(2s-\phi) + \sin 2\phi -$$

$$- 2K\sin(s-2\phi) - K^2\sin 2(s-\phi)\Big] -$$

$$- \frac{\gamma n}{256}(\gamma+1)(1-u_0)s^2\,\frac{\Big[\frac{K}{(1+u_0)^2} + \frac{1}{(1-u_0)^2}\Big]}{1-\cos s}\Big[\cos\phi + (K-1)\cos(s-\phi)$$

$$- \cos 2\phi - (K-1)\cos(s-2\phi) - K\cos(2s-\phi) + K\cos 2(s-\phi)\Big]$$

$$+ \frac{\gamma n(3-\gamma)Ks}{128(1-u_0)}\Big[\sin(s-\phi) - \sin(s-2\phi) + K\sin(2s-\phi) -$$

$$- K\sin 2(s-\phi)\Big] + \frac{\gamma n(3-\gamma)s(1-u_0)}{128(1+u_0)^2}\Big[-\sin\phi + K\sin(s-\phi)$$

$$+ \sin 2\phi - K\sin(s-2\phi)\Big] + \Big[3\left(\frac{2\gamma n}{\gamma-1}\right)^2 - 2M\Big]\left(\frac{\gamma-1}{8}\right)^2 s\Big[\sin 2\phi -$$

$$- 2K\sin(s-2\phi) - K^2\sin 2(s-\phi)\Big] + \Big[1-\gamma n\Big]\frac{\gamma n s}{16}\Big[\sin\phi -$$

$$- 2K\sin(s-\phi) - K^2\sin(2s-\phi)\Big]$$

$$Z_2 \equiv \gamma n(1-u_0^2)\Big\{-\frac{(\gamma+1)s^2}{128}\,\frac{\Big[K\left(\frac{1}{1+u_0}\right)^2 + \left(\frac{1}{1-u_0}\right)^2\Big]}{1-\cos s}\Big[(1-K)\sin(s-2\phi)$$

$$+ K\sin 2(s-\phi) + \sin 2\phi\Big] + \frac{3-\gamma}{64}s\Big[\frac{K}{(1-u_0)^2}\Big(\cos(s-2\phi) + K\cos 2(s-\phi)\Big)$$

$$+ \frac{\cos 2\phi + K\cos(s-2\phi)}{(1+u_0)^2}\Big]\Big\} - \gamma n s A\left[\cos 2(\theta-\phi) + K\cos 2(\theta+s-\phi)\right] -$$

$$-\left(\frac{\gamma n}{4}\right)^2 s\left[\cos\phi + 2K\cos(s-\phi) + K^2\cos(2s-\phi) - \cos 2\phi\right.$$

$$\left. - 2K\cos(s-2\phi) - K^2\cos 2(s-\phi)\right] -$$

$$- \frac{\gamma n(\gamma+1)(1-u_0)}{256} s^2 \frac{\left[\frac{K}{(1+u_0)^2} + \frac{1}{(1-u_0)^2}\right]}{(1-\cos s)}\left[-\sin\phi + (K-1)\sin(s-\phi)\right.$$

$$\left. + \sin 2\phi - (K-1)\sin(s-2\phi) - K\sin(2s-\phi) + K\sin 2(s-\phi)\right]$$

$$- \frac{\gamma n(3-\gamma)Ks}{128(1-u_0)}\left[\cos(s-\phi) - \cos(s-2\phi) + K\cos(2s-\phi)\right.$$

$$\left. - K\cos 2(s-\phi)\right] - \frac{\gamma n(1-u_0)(3-\gamma)s}{128(1+u_0)^2}\left[\cos\phi + \right.$$

$$\left. + K\cos(s-\phi) - \cos 2\phi - K\cos(s-2\phi)\right] +$$

$$+ \left[3\left(\frac{2\gamma n}{\gamma-1}\right)^2 - 2M\right]\left(\frac{\gamma-1}{8}\right)^2 s\left[\cos 2\phi + 2K\cos(s-2\phi) + K^2\cos 2(s-\phi)\right]$$

$$+ \left[1-\gamma n\right]\frac{\gamma n}{16} s\left[\cos\phi + 2K\cos(s-\phi) + K^2\cos(2s-\phi)\right]$$

$$Z_3 \equiv \frac{\gamma n(1-u_0^2)(\gamma+1)s^2}{64}\left[K\left(\frac{1}{1+u_0}\right)^2 + \left(\frac{1}{1-u_0}\right)^2\right](K-1)$$

$$+ \frac{\gamma n s}{16}(3-\gamma)K\sin s\,\frac{u_0}{1-u_0^2} + 2\left(\frac{\gamma n}{4}\right)^2 s\left[(1+K^2)\sin\phi + \right.$$

$$\left. + K\sin(s+\phi) - K\sin(s-\phi)\right] + \frac{\gamma n s}{8}(1-\gamma n)\left[(K^2+1)\sin\phi - \right.$$

$$\left. - K\sin(s-\phi) + K\sin(s+\phi)\right] -$$

$$-\frac{\gamma n}{128}(\gamma+1)(1-u_0)\,s^2\,\frac{\left[\frac{K}{(1+u_0)^2}+\frac{1}{(1-u_0)^2}\right]}{1-\cos s}\Big[K\cos(s-\phi)-(K-1)\cos s$$

$$-\cos(s+\phi)-(K-1)\cos\phi+(K-1)\Big]+\frac{\gamma n}{64}\,\frac{(3-\gamma)}{(1-u_0)}\,Ks\Big[-\sin s+$$

$$+K\sin\phi+\sin(s+\phi)\Big]+\frac{\gamma n}{64}(3-\gamma)\frac{1-u_0}{(1+u_0)^2}\,s\Big[\sin\phi-$$

$$-K\sin(s-\phi)+K\sin s\Big]$$

Now, the eighteenth term in R_3' may be determined

$$\left[(z_1+iz_2)e^{2i\psi_\beta}+(z_1-iz_2)e^{-2i\psi_\beta}+z_3\right]\frac{1-u_0^2}{2}\left[t_1-t_{1_{b_0}}\right]=$$

$$=(c_{18}+id_{18})e^{3i\psi_\beta}+(c_{18}-id_{18})e^{-3i\psi_\beta}+(g_{18}+ih_{18})e^{i\psi_\beta}$$

$$+(g_{18}-ih_{18})e^{-i\psi_\beta}$$

where

$$c_{18}\equiv-\frac{\gamma+1}{32}\,\frac{\left[K\frac{1-u_0}{1+u_0}+\frac{1+u_0}{1-u_0}\right]}{1-\cos s}\Big[z_1\big(1-\cos s-\cos\phi+\cos(s-\phi)\big)$$

$$-z_2\big(-\sin s+\sin\phi+\sin(s-\phi)\big)\Big]+$$

$$+\frac{3-\gamma}{16}\,\frac{K}{s}\,\frac{1+u_0}{1-u_0}\Big[z_2\big(\cos s-\cos(s-\phi)\big)+z_1\big(\sin s-\sin(s-\phi)\big)\Big]$$

$$+\frac{3-\gamma}{16}\,\frac{1}{s}\,\frac{1-u_0}{1+u_0}\Big[z_1\sin\phi+z_2(1-\cos\phi)\Big]$$

$$d_{18}\equiv-\frac{\gamma+1}{32}\,\frac{\left[K\frac{1-u_0}{1+u_0}+\frac{1+u_0}{1-u_0}\right]}{1-\cos s}\Big[z_2\big(1-\cos s-\cos\phi+\cos(s-\phi)\big)+$$

$$+z_1\big(-\sin s+\sin\phi+\sin(s-\phi)\big)\Big]-$$

$$-\frac{3-\gamma}{16}\;\frac{K}{S}\;\frac{1+u_0}{1-u_0}\left[Z_1\left(\cos s-\cos(s-\phi)\right)-Z_2\left(\sin s-\sin(s-\phi)\right)\right]$$

$$-\frac{3-\gamma}{16}\;\frac{1}{s}\;\frac{1-u_0}{1+u_0}\left[Z_1\left(1-\cos\phi\right)-Z_2\sin\phi\right]$$

$$q_{18}\equiv-\frac{\gamma+1}{32}\;\frac{\left[K\frac{1-u_0}{1+u_0}+\frac{1+u_0}{1-u_0}\right]}{1-\cos s}\left[Z_1\left(1-\cos s-\cos\phi+\cos(s-\phi)\right)\right.$$

$$\left.-Z_2\left(\sin s-\sin\phi-\sin(s-\phi)\right)\right]$$

$$+\frac{3-\gamma}{16}\;\frac{K}{s}\;\frac{1+u_0}{1-u_0}\left[Z_1\left(-\sin(s-\phi)+\sin s\right)+\right.$$

$$\left.+Z_2\left(\cos(s-\phi)-\cos s\right)\right]+\frac{3-\gamma}{16}\;\frac{1}{s}\;\frac{1-u_0}{1+u_0}\left[Z_1\sin\phi+\right.$$

$$\left.+Z_2\left(\cos\phi-1\right)\right]-\frac{\gamma+1}{32}\;\frac{\left[K\frac{1-u_0}{1+u_0}+\frac{1+u_0}{1-u_0}\right]}{1-\cos s}\,Z_3\left[1-\cos s-\right.$$

$$\left.-\cos\phi+\cos(s-\phi)\right]+\frac{3-\gamma}{16}\;\frac{K}{S}\;\frac{1+u_0}{1-u_0}\,Z_3\left[\sin s-\right.$$

$$\left.-\sin(s-\phi)\right]+\frac{3-\gamma}{16}\;\frac{1}{s}\;\frac{1-u_0}{1+u_0}\;Z_3\sin\phi$$

$$h_{18}\equiv-\frac{\gamma+1}{32}\;\frac{\left[K\frac{1-u_0}{1+u_0}+\frac{1+u_0}{1-u_0}\right]}{1-\cos s}\left[Z_1\left(\sin s-\sin\phi-\sin(s-\phi)\right)\right.$$

$$\left.+Z_2\left(1-\cos s-\cos\phi+\cos(s-\phi)\right)\right]+$$

$$+\frac{3-\gamma}{16}\;\frac{K}{S}\;\frac{1+u_0}{1-u_0}\left[Z_1\left(\cos s-\cos(s-\phi)\right)+Z_2\left(\sin s-\right.\right.$$

$$\left.\left.-\sin(s-\phi)\right)\right]+\frac{3-\gamma}{16}\;\frac{1}{s}\;\frac{1-u_0}{1+u_0}\left[Z_1\left(1-\cos\phi\right)+Z_2\sin\phi\right]$$

$$-\frac{\gamma+1}{32}\ \frac{\left[K\frac{1-u_0}{1+u_0}+\frac{1+u_0}{1-u_0}\right]}{1-\cos s}\ Z_3\left[\sin\phi-\sin s+\sin(s-\phi)\right]$$

$$-\frac{3-\gamma}{16}\ \frac{K}{s}\ \frac{1+u_0}{1-u_0}\ Z_3\left[\cos s-\cos(s-\phi)\right]-$$

$$-\frac{3-\gamma}{16}\ \frac{1}{s}\ \frac{1-u_0}{1+u_0}\ Z_3\ (1-\cos\phi)$$

Note that $m_{18} = 1$.

Equations (39) and (41) are used to substitute in the nineteenth term with the result

$$-\left(\frac{2\gamma n}{\gamma-1}\right)^2\left(\frac{1-u_0^2}{2}\right)s\,\frac{da}{d\psi_\beta}{}_{,b_0}\int_{\psi_\beta-\phi}^{\psi_\beta}a_1\,\frac{dt_1}{d\psi_\beta}\,d\psi_\beta' = m_{19}\left\{(c_{19}+id_{19})e^{3i\psi_\beta}\right.$$

$$\left.+(c_{19}-id_{19})e^{-3i\psi_\beta}+(q_{19}+ih_{19})e^{i\psi_\beta}+(q_{19}-ih_{19})e^{-i\psi_\beta}\right\}$$

where

$$m_{19}\equiv-\left(\frac{2\gamma n}{\gamma-1}\right)^2\left(\frac{1-u_0^2}{2}\right)$$

$$c_{19}\equiv\frac{(\gamma+1)(\gamma-1)^2 s}{2048}\ \frac{\left[K\left(\frac{1}{1+u_0}\right)^2+\left(\frac{1}{1-u_0}\right)^2\right]}{1-\cos s}\left[\sin\phi+(2K-1)\sin(s-\phi)\right.$$

$$+(K^2-2K)\sin(2s-\phi)-K^2\sin(3s-\phi)+\sin 3\phi-(2K-1)\sin(s-3\phi)$$

$$\left.-(K^2-2K)\sin(2s-3\phi)+K^2\sin 3(s-\phi)\right]+$$

$$+\frac{(3-\gamma)(\gamma-1)^2}{1024}\left\{\frac{K}{(1-u_0)^2}\left[\cos(s-\phi)+2K\cos(2s-\phi)+K^2\cos(3s-\phi)\right.\right.$$

$$- \cos (s-3\phi) - 2K \cos (2s-3\phi) - K^2 \cos 3(s-\phi)\Big] +$$

$$+ \frac{1}{(1+u_0)^2}\Big[\cos\phi + 2K \cos (s-\phi) + K^2 \cos (2s-\phi) - \cos 3\phi$$

$$- 2K \cos (s-3\phi) - K^2 \cos (2s-3\phi)\Big]\Big\}$$

$$d_{19} \equiv \frac{(\gamma+1)(\gamma-1)^2 s}{2048} \quad \frac{\left[K\left(\frac{1}{1+u_0}\right)^2 + \left(\frac{1}{1-u_0}\right)^2\right]}{1-\cos s}\Big[\cos 3\phi + (2K-1)\cos(s-3\phi)$$

$$+ (K^2-2K)\cos(2s-3\phi) - K^2 \cos 3(s-\phi) - \cos\phi - (2K-1)\cos(s-\phi)$$

$$-(K^2-2K)\cos(2s-\phi) + K^2 \cos(3s-\phi)\Big] +$$

$$+ \frac{(3-\gamma)(\gamma-1)^2}{1024}\Big\{\frac{K}{(1-u_0)^2}\Big[\sin(s-\phi) + 2K\sin(2s-\phi) +$$

$$+ K^2 \sin(3s-\phi) - \sin(s-3\phi) - 2K\sin(2s-3\phi) - K^2 \sin 3(s-\phi)\Big]$$

$$+ \frac{1}{(1+u_0)^2}\Big[-\sin\phi + 2K\sin(s-\phi) + K^2\sin(2s-\phi) + \sin 3\phi -$$

$$- 2K\sin(s-3\phi) - K^2\sin(2s-3\phi)\Big]\Big\}$$

$$g_{19} \equiv \frac{(\gamma+1)(\gamma-1)^2 s}{1024} \quad \frac{\left[K\left(\frac{1}{1+u_0}\right)^2 + \left(\frac{1}{1-u_0}\right)^2\right]}{1-\cos s}\Big[\Big((\cos s-1)(1+K^2 +$$

$$+ 2K\cos s)\sin\phi + \phi\,(\cos\phi - \cos(s-\phi)\Big)(1+K^2+2K\cos s)$$

$$+ \phi\Big(\cos(s+\phi) + (2K-1)\cos\phi - (2K-K^2)\cos(s-\phi) -$$

$$- K^2\cos(2s-\phi)\Big)\Big] + \frac{(3-\gamma)(\gamma-1)^2}{512}\Big\{\frac{K}{(1-u_0)^2}\Big[(1+K^2+2K\cos s)\sin\phi \sin s$$

$$+2\phi\,(K+\cos s)\Big(\sin\phi - K\sin(s-\phi)\Big)\Big]+$$

$$+\left(\frac{1}{1+u_0}\right)^2\Big[2\phi\,(1+K\cos s)\Big(\sin\phi - K\sin(s-\phi)\Big)\Big]\Big\}$$

$$h_{19} \equiv \frac{(\gamma+1)(\gamma-1)^2 s}{1024}\;\frac{\left[K\left(\frac{1}{1+u_0}\right)^2+\left(\frac{1}{1-u_0}\right)^2\right]}{1-\cos s}\Big[\sin s\,(1+K^2+2K\cos s)\sin\phi -$$

$$-\phi\,(1+K^2+2K\cos s)\Big(\sin\phi+\sin(s-\phi)\Big)+$$

$$+\phi\Big(-\sin(s+\phi)-(2K-1)\sin\phi-(2K-K^2)\sin(s-\phi)-$$

$$-K^2\sin(2s-\phi)\Big)\Big]+\frac{(3-\gamma)(\gamma-1)^2}{512}\left\{\frac{K}{(1-u_0)^2}\Big[-(1+K^2+2K\cos s)\sin\phi\cos s\right.$$

$$+2\phi\,(K+\cos s)\Big(\cos\phi+K\cos(s-\phi)\Big)\Big]+\left(\frac{1}{1+u_0}\right)^2\Big[-(1+K^2+2K\cos s)\sin\phi$$

$$\left.+2\phi\,(1+K\cos s)\,(\cos\phi+K\cos(s-\phi))\Big]\right\}$$

By means of similar substitution, the twentieth term is found

$$\left(\frac{2\gamma n}{\gamma-1}\right)^2\left(\frac{1-u_0^2}{2}\right)s\;\frac{da}{d\psi_\beta}\Big|_{b_0}\frac{dt}{d\psi_\beta}\Big|_{b_0}\int_{\psi_\beta-\phi}^{\psi_\beta} a_1\,d\psi'_\beta = m_{20}\Big\{(c_{20}+id_{20})e^{3i\psi_\beta}$$

$$+(c_{20}-id_{20})\,e^{-3i\psi_\beta}+(g_{20}+ih_{20})\,e^{i\psi_\beta}+(g_{20}-ih_{20})\,e^{-i\psi_\beta}\Big\}$$

where

$$m_{20} \equiv \left(\frac{2\gamma n}{\gamma-1}\right)^2\left(\frac{1-u_0^2}{2}\right)$$

$$c_{20} \equiv \frac{(\gamma-1)^2(\gamma+1)s}{1024} \; \frac{\left[K\left(\frac{1}{1+u_0}\right)^2 + \left(\frac{1}{1-u_0}\right)^2\right]}{1-\cos s} \Big[(1-2K) \sin (s-3\phi) +$$

$$+ (2K-K^2) \sin (2s-3\phi) + K^2 \sin 3(s-\phi) + \sin 3\phi - (1-2K) \sin (s-2\phi)$$

$$- (2K-K^2) \sin 2(s-\phi) - K^2 \sin (3s-2\phi) - \sin 2\phi \Big] +$$

$$+ \frac{(3-\gamma)(\gamma-1)^2}{512} \left[\frac{K}{(1-u_0)^2}\Big(\cos (s-2\phi) + 2K \cos 2(s-\phi) +\right.$$

$$+ K^2 \cos (3s-2\phi) - \cos (s-3\phi) - 2K \cos (2s-3\phi)$$

$$- K^2 \cos 3(s-\phi)\Big) + \left(\frac{1}{1+u_0}\right)^2 \Big(\cos 2\phi + 2K \cos (s-2\phi)$$

$$\left. + K^2 \cos 2(s-\phi) - \cos 3\phi - 2K \cos (s-3\phi) - K^2 \cos (2s-3\phi)\Big)\right]$$

$$d_{20} \equiv \frac{(\gamma+1)(\gamma-1)^2 s}{1024} \; \frac{\left[K\left(\frac{1}{1+u_0}\right)^2 + \left(\frac{1}{1-u_0}\right)^2\right]}{1-\cos s} \Big[(1-2K) \cos (s-2\phi) +$$

$$+ (2K-K^2) \cos 2(s-\phi) + K^2 \cos (3s-2\phi) - \cos 2\phi -$$

$$- (1-2K) \cos (s-3\phi) - (2K-K^2) \cos (2s-3\phi) - K^2 \cos 3(s-\phi) +$$

$$+ \cos 3\phi\Big] + \frac{(3-\gamma)(\gamma-1)^2}{512} \left[\frac{K}{(1-u_0)^2}\Big(\sin (s-2\phi) + 2K \sin 2(s-\phi)\right.$$

$$+ K^2 \sin (3s-2\phi) - \sin (s-3\phi) - 2K \sin (2s-3\phi) -$$

$$- K^2 \sin 3(s-\phi)\Big) + \left(\frac{1}{1+u_0}\right)^2 \Big(- \sin 2\phi + 2K \sin (s-2\phi)$$

$$+ K^2 \sin 2(s-\phi) + \sin 3\phi - 2K \sin (s-3\phi) - K^2 \sin (2s-3\phi)\Big)\Big]$$

$$q_{20} \equiv \frac{(\gamma+1)(\gamma-1)^2 s}{1024} \quad \frac{\left[K\left(\frac{1}{1+u_0}\right)^2 + \left(\frac{1}{1-u_0}\right)^2\right]}{1-\cos s} \Big\{\Big[(K^2-2K)\sin s -$$

$$- K^2 \sin 2s - \sin s - (1-2K)\sin\phi + (2K-K^2)\sin(s-\phi)$$

$$+ K^2 \sin (2s-\phi) + \sin (s+\phi)\Big] - 2(1+K^2+2K\cos s)(1-\cos\phi)(\sin\phi +$$

$$+ \sin (s-\phi)\Big)\Big\} + \frac{(3-\gamma)(\gamma-1)^2}{512} \Big\{\frac{K}{(1-u_0)^2}\Big[(1+K^2)\cos s + 2K -$$

$$- \cos (s+\phi) - 2K\cos\phi - K^2 \cos (s-\phi)\Big] +$$

$$+ \left(\frac{1}{1+u_0}\right)^2 \Big[1 + 2K\cos s + K^2 \cos 2s - \cos\phi - 2K\cos(s-\phi) -$$

$$- K^2 \cos (2s-\phi)\Big] + 2(1+K^2+2K\cos s)(1-\cos\phi)\Big[\frac{\cos\phi}{(1+u_0)^2} +$$

$$+ \frac{K\cos(s-\phi)}{(1-u_0)^2}\Big]\Big\}$$

$$h_{20} \equiv \frac{(\gamma+1)(\gamma-1)^2 s}{1024} \quad \frac{\left[K\left(\frac{1}{1+u_0}\right)^2 + \left(\frac{1}{1-u_0}\right)^2\right]}{1-\cos s} \Big\{\Big[1 - 2K + (2K-K^2-1)\cos s$$

$$+ K^2 \cos 2s - (1-2K)\cos\phi - (2K-K^2)\cos(s-\phi) - K^2\cos(2s-\phi)$$

$$+ \cos(s-\phi)\Big] + 2(1+K^2+2K\cos s)(1-\cos\phi)\Big[\cos(s-\phi) - \cos\phi\Big]\Big\}$$

$$+ \frac{(3-\gamma)(\gamma-1)^2}{512} \Big\{\frac{K}{(1-u_0)^2}\Big[(K^2-1)\sin s + \sin(s+\phi) + 2K\sin\phi$$

$$-K^2 \sin(s-\phi)] + \left(\frac{1}{1+u_0}\right)^2 [2K \sin s + K^2 \sin 2s + \sin\phi -$$

$$-2K \sin(s-\phi) - K^2 \sin(2s-\phi)] +$$

$$+2(1+K^2+2K\cos s)(1-\cos\phi)\left[\frac{K\sin(s-\phi)}{(1-u_0)^2} - \frac{\sin\phi}{(1+u_0)^2}\right]\Big\}$$

The final term in R_3' to be evaluated is the following

$$-\frac{\gamma n}{\gamma-1}(1-u_0^2)\, s\, \frac{da}{d\psi_\beta}{}'b_0 \left[t_2 - t_{2b_0}\right]$$

As a preliminary step, the expression $\left[t_2 - t_{2b_0}\right]$ is evaluated to be the following by means of Equation (50)

$$\left[t_2(\psi_\beta,\psi_\beta) - t_2(\psi_\beta-\phi, \psi_\beta-\phi)\right] = Z_4 + (Z_5 + iZ_6)e^{2i\psi_\beta} +$$

$$+ (Z_5 - iZ_6)e^{-2i\psi_\beta}$$

where the following definitions are applied

$$Z_4 \equiv \frac{\phi}{s}\Big\{V_2 + \frac{3-\gamma}{4}\left[\left(\frac{1}{1+u_0}\right)^2 + K\left(\frac{1}{1-u_0}\right)^2\right] C_r +$$

$$+\frac{3-\gamma}{8}\, s\left[\frac{1-u_0}{1+u_0}\ \frac{\sin s}{1-\cos s}\ \frac{\gamma+1}{16}\left(\frac{K}{(1+u_0)^2} + \frac{1}{(1-u_0)^2}\right) +\right.$$

$$\left.+\frac{1+u_0}{1-u_0} K \frac{\sin 2s - \sin s}{1-\cos s}\ \frac{\gamma+1}{16}\left(\frac{K}{(1+u_0)^2} + \frac{1}{(1-u_0)^2}\right)\right] +$$

$$+\frac{3-\gamma}{32}\left[\left(\frac{\gamma+1}{4(1-u_0^2)} + \frac{3-\gamma}{(1-u_0)^3}\right)K^2 + \left(\frac{\gamma+1}{4(1-u_0^2)} + \frac{3-\gamma}{(1+u_0)^3}\right)\right]\Big\}$$

$$Z_5 \equiv \frac{r_1 A\left(\cos 2\theta - \cos 2(s+\theta) - \cos 2(\theta-\phi) + \cos 2(s+\theta-\phi)\right)}{2(1-\cos 2s)}$$

$$+ \frac{r_2\left(1-\cos 2s - \cos 2\phi + \cos 2(s-\phi)\right)}{8(1-\cos 2s)}$$

$$+ \frac{r_3 K\left(\sin s - \sin 3s + \sin 2s - \sin (s-2\phi)\right)}{8s(1-\cos 2s)} +$$

$$+ \frac{r_3 K\left(-\sin 2\phi + \sin (3s-2\phi) - \sin 2(s-\phi)\right)}{8s(1-\cos 2s)}$$

$$+ \frac{r_4\left(2\sin s - \sin 2s + \sin 2\phi - \sin (s+2\phi) + \sin 2(s-\phi) - \sin (s-2\phi)\right)}{8s(1-\cos 2s)}$$

$$+ \frac{3-\gamma}{8}\left\{\left(\frac{1}{1+u_0}\right)^2 \frac{A}{s}\left(\sin 2\theta - \sin 2(\theta-\phi)\right) + \right.$$

$$+ \left(\frac{1}{1-u_0}\right)^2 \frac{AK}{s}\left(\sin 2(\theta+s) - \sin 2(\theta+s-\phi)\right) -$$

$$- \frac{\gamma+1}{64}\ \frac{1-u_0}{1+u_0}\left[\frac{K}{(1+u_0)^2} + \frac{1}{(1-u_0)^2}\right]\left(1 - \frac{\cos 2\phi - \cos (s+2\phi)}{1-\cos s}\right.$$

$$+ \frac{\gamma+1}{64}\ \frac{1+u_0}{1-u_0} K\left[\frac{K}{(1+u_0)^2} + \frac{1}{(1-u_0)^2}\right]\left(1 - \frac{\cos 2\phi - \cos (s-2\phi)}{1-\cos s}\right)$$

$$+ \frac{1}{16}\ \frac{K^2}{s}\left[\frac{\gamma+1}{4(1-u_0^2)} + \frac{3-\gamma}{(1-u_0)^3}\right]\left(\sin 2s - \sin 2(s-\phi)\right) +$$

$$\left. + \frac{1}{16s}\left[\frac{\gamma+1}{4(1-u_0^2)} + \frac{3-\gamma}{(1+u_0)^3}\right]\sin 2\phi\right\}$$

$$Z_6 \equiv \frac{r_1 A\left(\sin 2\theta - \sin 2(s+\theta) - \sin 2(\theta-\phi) + \sin 2(s+\theta-\phi)\right)}{2(1-\cos 2s)}$$

$$+ \frac{r_2\left(-\sin 2s - \sin 2\phi + \sin 2(s-\phi)\right)}{8(1-\cos 2s)}$$

$$- \frac{r_3 K\left(\cos s - 1 - \cos 3s + \cos 2s - \cos(s-2\phi) + \cos 2\phi\right)}{8s(1-\cos 2s)}$$

$$- \frac{r_3 K\left(\cos(3s-2\phi) - \cos 2(s-\phi)\right)}{8s(1-\cos 2s)}$$

$$- \frac{r_4\left(1 - \cos 2s - \cos 2\phi + \cos(s+2\phi) + \cos 2(s-\phi) - \cos(s-2\phi)\right)}{8s(1-\cos 2s)}$$

$$+ \frac{3-\gamma}{8}\left\{\left(\frac{1}{1+u_0}\right)^2 \frac{A}{s}\left(\cos 2(\theta-\phi) - \cos 2\theta\right) + \right.$$

$$\left. + \left(\frac{1}{1-u_0}\right)^2 \frac{AK}{s}\left(\cos 2(\theta+s-\phi) - \cos 2(\theta+s)\right)\right\}$$

$$+ \frac{\gamma+1}{64}\,\frac{1-u_0}{1+u_0}\left[\frac{K}{(1+u_0)^2} + \frac{1}{(1-u_0)^2}\right]\frac{\left(-\sin 2\phi + \sin(s+2\phi) - \sin s\right)}{(1-\cos s)}$$

$$+ \frac{\gamma+1}{64}\,\frac{1+u_0}{1-u_0}\,K\left[\frac{K}{(1+u_0)^2} + \frac{1}{(1-u_0)^2}\right]\frac{\left(-\sin s + \sin 2\phi + \sin(s-2\phi)\right)}{1-\cos s}$$

$$+ \frac{K^2}{16s}\left[\frac{\gamma+1}{4(1-u_0^2)} + \frac{3-\gamma}{(1-u_0)^3}\right]\left(\cos 2(s-\phi) - \cos 2s\right) +$$

$$+\frac{1}{16s}\left[\frac{\gamma+1}{4(1-u_o^2)}+\frac{3-\gamma}{(1+u_o)^3}\right](\cos 2\phi-1)\Big\}$$

Now, it is found that

$$-\frac{\gamma n}{\gamma-1}(1-u_o^2)s\frac{da}{d\psi_\beta}\,b_o\left[t_2-t_{2_{b_o}}\right]=m_{21}\Big\{(c_{21}+id_{21})e^{3i\psi_\beta}$$

$$+(c_{21}-id_{21})e^{-3i\psi_\beta}+(g_{21}+ih_{21})e^{i\psi_\beta}+(g_{21}-ih_{21})e^{-i\psi_\beta}\Big\}$$

where

$$m_{21}\equiv-\frac{\gamma n}{\gamma-1}(1-u_o^2)$$

$$c_{21}\equiv\frac{\gamma-1}{8}s\Big\{-Z_6\big(\cos\phi+K\cos(s-\phi)\big)+Z_5\big(\sin\phi-K\sin(s-\phi)\big)\Big\}$$

$$d_{21}\equiv\frac{\gamma-1}{8}s\Big\{Z_5\big(\cos\phi+K\cos(s-\phi)\big)+Z_6\big(\sin\phi-K\sin(s-\phi)\big)\Big\}$$

$$g_{21}\equiv\frac{\gamma-1}{8}s\Big\{Z_4\big(\sin\phi-K\sin(s-\phi)\big)+Z_5\big(\sin\phi-K\sin(s-\phi)\big)$$

$$+Z_6\big(\cos\phi+K\cos(s-\phi)\big)\Big\}$$

$$h_{21}\equiv\frac{\gamma-1}{8}s\Big\{Z_4\big(\cos\phi+K\cos(s-\phi)\big)-Z_5\big(\cos\phi+K\cos(s-\phi)\big)$$

$$+Z_6\big(\sin\phi-K\sin(s-\phi)\big)\Big\}$$

It is convenient to define

$$b_r\equiv\sum_{i=1}^{21}m_i\,c_i$$

$$b_i\equiv\sum_{i=1}^{21}m_i\,d_i$$

$$l_r\equiv\sum_{i=1}^{21}m_i\,g_i$$

$$l_i\equiv\sum_{i=1}^{21}m_i\,h_i$$

such that

$$R_3' = (b_r + ib_i)e^{3i\psi_\beta} + (b_r - ib_i)e^{-3i\psi_\beta} + (l_r + il_i)e^{i\psi_\beta} + (l_r - il_i)e^{-i\psi_\beta}$$

This result may be substituted in Equation (51) while Equation (39) is used to substitute for ρ_1 and its derivative in Equation (51) so that the final result is

$$\begin{aligned}
&K[1 + u_0(1-\gamma n^{(0)})]\,Q_3\,(s^{(0)} + \psi_\beta) - [1 - u_0(1-\gamma n^{(0)})]\,Q_3\,(\psi_\beta) \\
&+ \gamma n^{(0)} u_0 [K Q_3 (s^{(0)} + \psi_\beta - \phi^{(0)}) + Q_3\,(\psi_\beta - \phi^{(0)})] = \\
&= 2u_0 [(b_r + ib_i)e^{3i\psi_\beta} + (b_r - ib_i)e^{-3i\psi_\beta} + (l_r + il_i)e^{i\psi_\beta} + \\
&+ (l_r - il_i)e^{-i\psi_\beta}] - n^{(2)}[(\omega_1 + iy_1)e^{i\psi_\beta} + (\omega_1 - iy_1)e^{-i\psi_\beta}] \\
&- s^{(2)}[(\omega_2 + iy_2)e^{i\psi_\beta} + (\omega_2 - iy_2)e^{-i\psi_\beta}] - \\
&- \phi^{(2)}[(\omega_3 + iy_3)e^{i\psi_\beta} + (\omega_3 - iy_3)e^{-i\psi_\beta}]
\end{aligned} \tag{52}$$

where the following definitions are used

$$\omega_1 \equiv \frac{\gamma u_0}{2}\left[-K\cos s^{(0)} - 1 + K\cos(s^{(0)} - \phi^{(0)}) + \cos\phi^{(0)}\right]$$

$$y_1 \equiv \frac{\gamma u_0}{2}\left[-K\sin s^{(0)} + K\sin(s^{(0)} - \phi^{(0)}) - \sin\phi^{(0)}\right]$$

$$\omega_2 \equiv \frac{-K}{2}\left[\left(1 + u_0(1-\gamma n^{(0)})\right)\sin s^{(0)} + \gamma n^{(0)} u_0 \sin(s^{(0)} - \phi^{(0)})\right]$$

$$y_2 \equiv \frac{K}{2}\left[\left(1 + u_0(1-\gamma n^{(0)})\right)\cos s^{(0)} + \gamma n^{(0)} u_0 \cos(s^{(0)} - \phi^{(0)})\right]$$

$$\omega_3 \equiv \frac{\gamma n^{(0)} u_0}{2}\left[K\sin\left(s^{(0)} - \phi^{(0)}\right) - \sin\phi^{(0)}\right]$$

$$y_3 \equiv -\frac{\gamma n^{(0)} u_o}{2}\left[K\cos(s^{(0)}-\phi^{(0)})+\cos\phi^{(0)}\right]$$

It is seen that the inhomogeneous terms contain the first and third harmonics. Since the first harmonic is contained solely in the Q_1 term, the Q_3 term must contain only a third harmonic term. Therefore, the coefficients of the first harmonic terms in the inhomogeneous part must sum to zero and a complex relationship for $n^{(2)}$, $\phi^{(2)}$, and $s^{(2)}$ is obtained. The solution is assumed to be of the form

$$Q_3 = (B_r + iB_i)e^{3i\psi_\beta} + (B_r - iB_i)e^{-3i\psi_\beta}$$

where B_r and B_i are to be determined. If the definitions are made that

$$\xi_r \equiv K[1+u_o(1-\gamma n^{(0)})]\cos 3s^{(0)} - [1-u_o(1-\gamma n^{(0)})] +$$

$$+\gamma n^{(0)}u_o\left[K\cos 3(s^{(0)}-\phi^{(0)})+\cos 3\phi^{(0)}\right]$$

$$\xi_i \equiv K[1+u_o(1-\gamma n^{(0)})]\sin 3s^{(0)} + \gamma n^{(0)}u_o\left[K\sin 3(s^{(0)}-\phi^{(0)})-\sin 3\phi^{(0)}\right]$$

it is found by substitution in Equation (52) that

$$(\xi_r + i\xi_i)(B_r+iB_i)e^{3i\psi_\beta} + (\xi_r - i\xi_i)(B_r-iB_i)e^{-3i\psi_\beta} =$$

$$= 2u_o(b_r+ib_i)e^{3i\psi_\beta} + 2u_o(b_r-ib_i)e^{-3i\psi_\beta} +$$

$$+\left[2u_o(l_r+il_i)-(\omega_1+iy_1)n^{(2)}-(\omega_2+iy_2)s^{(2)}-(\omega_3+iy_3)\phi^{(2)}\right]e^{i\psi_\beta}$$

$$+\left[2u_o(l_r-il_i)-(\omega_1-iy_1)n^{(2)}-(\omega_2-iy_2)s^{(2)}-(\omega_3-iy_3)\phi^{(2)}\right]e^{-i\psi_\beta}$$

Separation of the coefficients of $e^{3i\psi_\beta}$, $e^{-3i\psi_\beta}$, $e^{i\psi_\beta}$, and $e^{-i\psi_\beta}$ and further separation of the real and imaginary parts of these coefficients lead to the following relationships

$$\xi_r B_r - \xi_i B_i = 2u_o b_r \tag{53a}$$

$$\xi_r B_i + \xi_i B_r = 2u_o b_i \tag{53b}$$

$$\omega_1 n^{(2)} + \omega_2 s^{(2)} + \omega_3 \phi^{(2)} = 2u_o l_r \tag{53c}$$

$$y_1 n^{(2)} + y_2 s^{(2)} + y_3 \phi^{(2)} = 2u_o l_i \tag{53d}$$

Equations (53a) and (53b) are solved to obtain

$$B_r = 2u_o \left[\frac{\xi_i b_i + \xi_r b_r}{\xi_r^2 + \xi_i^2} \right]$$

$$B_i = 2u_o \left[\frac{\xi_r b_i - \xi_i b_r}{\xi_r^2 + \xi_i^2} \right]$$

It is convenient to define

$$\Delta \equiv \frac{1}{3} \arctan \left(\frac{B_i}{B_r} \right)$$

$$B \equiv \frac{B_r}{\cos 3\Delta} = \frac{B_i}{\sin 3\Delta} \tag{54}$$

such that

$$Q_3 = B \left[e^{3i(\psi_\alpha + \Delta)} + e^{-3i(\psi_\alpha + \Delta)} \right]$$

$$= 2B \cos 3 (\psi_\alpha + \Delta) \tag{55a}$$

It follows from Equations (11) and (12) that

$$P_3 = 2BK \cos 3 (\psi_\beta + S + \Delta)$$

$$u_3 = B \left[K \cos 3 (\psi_\beta + S + \Delta) - \cos 3 (\psi_\alpha + \Delta) \right]$$

$$a_3 = \frac{\gamma - 1}{2} B \left[K \cos 3 (\psi_\beta + S + \Delta) + \cos 3 (\psi_\alpha + \Delta) \right] \tag{55b}$$

The third order contribution to the waveform is the addition of the third harmonic with a phase.

Before solving Equations (53c) and (53d), it is convenient to transform from ϕ back to τ_0 . It is seen that

$$\tau_0 = \left(\frac{2}{1-u_0^2}\right) b_0 = \left(\frac{2}{1-u_0^2}\right)\frac{\phi}{s}$$

Substitution of the series (42) and separation according to powers in ϵ shows that

$$\phi^{(0)} = \frac{1-u_0^2}{2} \; s^{(0)} \tau_0^{(0)}$$

$$\phi^{(2)} = \frac{1-u_0^2}{2} s^{(0)} \tau_0^{(2)} + \frac{\phi^{(0)}}{s^{(0)}} \; s^{(2)}$$

The above relations are combined with Equations (53) to yield

$$\omega_1 n^{(2)} + \omega_2' \; s^{(2)} + \omega_3' \; \tau_0^{(2)} = 2 u_0 \ell_r$$

$$y_1 n^{(2)} + y_2' \; s^{(2)} + y_3' \; \tau_0^{(2)} = 2 u_0 \; \ell_i \tag{56}$$

where the following definitions have been made

$$\omega_2' \equiv \omega_2 + \omega_3 \; \frac{\phi^{(0)}}{s^{(0)}}$$

$$y_2' \equiv y_2 + y_3 \frac{\phi^{(0)}}{s^{(0)}}$$

$$\omega_3' \equiv \omega_3 \left(\frac{1-u_0^2}{2}\right) s^{(0)}$$

$$y_3' \equiv y_3 \left(\frac{1-u_0^2}{2}\right) s^{(0)}$$

Elimination of $s^{(2)}$ from Equations (56) has the result

$$[\omega_1 y_2' - y_1 \omega_2'] \, n^{(2)} + [\omega_3' y_2' - y_3' \omega_2'] \, \tau_0^{(2)} =$$

$$= 2 u_0 \, [\ell_r y_2' - \omega_2' \; \ell_i] \tag{57}$$

Now, if n' and τ_o' give the displacement from the neutral line in a τ_o vs. n parameter plot necessary to produce a periodic solution with an amplitude of ϵ for the first harmonic, it follows that

$$\tau_o' = \epsilon^2 \tau_o^{(2)} + O(\epsilon^3)$$
$$n' = \epsilon^2 n^{(2)} + O(\epsilon^3)$$

If $\tau_o^{(2)}$ and $n^{(2)}$ are not too large, this means a small displacement produces a finite amplitude oscillation. These relations may be substituted into Equation (57) to obtain the relationship between displacement from the neutral line and amplitude of oscillation. The result is

$$\frac{[\omega_1 y_2' - y_1 \omega_2']n' + [\omega_3' y_2' - y_3' \omega_2']\tau_o'}{\sqrt{[\omega_1 y_2' - y_1 \omega_2']^2 + [\omega_3' y_2' - y_3' \omega_2']^2}} = \frac{2u_o[\ell_r y_2' - \omega_2' \ell_i]\epsilon^2}{\sqrt{[\omega_1 y_2' - y_1 \omega_2']^2 + [\omega_3' y_2' - y_3' \omega_2']^2}} \tag{58}$$

The error in the above relationship is of order ϵ^3 and considered negligible. The coefficients of n' and τ_o' vary with position along the neutral line; i.e., they depend upon $n^{(o)}$, $\tau_o^{(o)}$, etc., so that the displacement has been normalized by the square root of the sum of the squares of the coefficients. If the left-hand side of Equation (58) is held constant for all points along the neutral line, it is implied that the magnitude of the displacement is constant in a direction normal to the curve. The direction, however, may be inward or outward. Therefore, depending upon the value of the factor

$$D \equiv \frac{2u_0\left[l_r \psi_2' - \omega_2' l_i\right]}{\sqrt{\left[\omega_1 \psi_2' - \psi_1 \omega_2'\right]^2 + \left[\omega_3' \psi_2' - \psi_3' \omega_2'\right]^2}} \tag{59}$$

a certain amplitude ϵ is obtained for the given displacement. If the displacement is held constant in magnitude along the neutral line, the amplitude will vary inversely with the square root of the above quantity D. The direction (inward or outward) of the displacement depends upon the sign of this quantity D and, of course, the signs of the coefficients of n' and τ'.

Another approach involves keeping the amplitude ϵ constant and determining the magnitude of the normal displacement necessary to obtain a periodic solution of this amplitude. This displacement varies directly proportionally to D along the neutral line.

If a displacement is normal to the $\tau_0^{(0)}$ vs. $n^{(0)}$ curve, it follows from inspection of Equation (57) that the components of the displacement in the n-direction and in the τ_0 -direction are related as follows

$$\frac{n'}{\tau_0'} = \frac{n^{(2)}}{\tau_0^{(2)}} = \frac{\left[\omega_1 \psi_2' - \psi_1 \omega_2'\right]}{\left[\omega_3' \psi_2' - \psi_3' \omega_2'\right]}$$

This may be combined with Equation (57) to obtain the value of $n^{(2)}$ for a normal displacement. The result is

$$n^{(2)} = \frac{2u_0\left[l_r \psi_2' - \omega_2' l_i\right]\left[\omega_1 \psi_2' - \psi_1 \omega_2'\right]}{\left[\omega_1 \psi_2' - \psi_1 \omega_2'\right]^2 + \left[\omega_3' \psi_2' - \psi_3' \omega_2'\right]^2}$$

This result may be combined with Equations (56) and solved for the value of $s^{(2)}$ for a normal displacement to the $\tau_0^{(0)}$ vs. $n^{(0)}$ curve in the τ vs. n plane. The result is as follows:

$$S^{(2)} = 2u_0 \left\{ \frac{[\psi_3' \omega_1 - \omega_3' \psi_1]}{[\omega_3' \psi_2' - \omega_2' \psi_3']} \quad \frac{[\omega_1 \psi_2' - \psi_1 \omega_2'][l_r \psi_2' - \omega_2' l_i]}{[\omega_1 \psi_2' - \psi_1 \omega_2']^2 + [\omega_3' \psi_2' - \psi_3' \omega_2']^2} - \frac{[\psi_3' l_r - \omega_3' l_i]}{[\omega_3' \psi_2' - \omega_2' \psi_3']} \right\}$$

Note that the factors appearing in Equation (58) depend upon $n^{(0)}$ and $\tau_0^{(0)}$. If the zero order parameters contain errors of order u_0^2, so will the perturbation coefficients $\tau_0^{(2)}$ and $n^{(2)}$. If $\tau_0^{(0)}$ and $n^{(0)}$ are exact, so are $\tau_0^{(2)}$ and $n^{(2)}$.

Equation (58) shows that for small ϵ (i.e., ϵ^3 negligible compared to ϵ^2), the curve representing displacement as a function of amplitude is a parabola which passes through the origin. In other words, the displacement goes to zero as the amplitude squared goes to zero. The shape of the parabola varies from point to point along the neutral line as the factor D varies.

A three-dimensional plot of τ_0 vs. n vs. ϵ may be constructed for a small range of $\epsilon \doteq 0$. A surface would be obtained whose intersection with the $\epsilon = 0$ plane gives the $\tau_0^{(0)}$ vs. $n^{(0)}$ curves as found originally by Crocco. The intersection of this surface with a plane perpendicular to the $\epsilon = 0$ plane would give a parabola. This surface represents the locus of points where periodic solutions are found. The stability of these periodic solutions still must be determined.

The coefficients x_3 and t_3 need not be determined since a third order correction in the coordinates is only consistent when the flow properties are determined to fourth order accuracy.

STABILITY OF THE PERIODIC SOLUTIONS

The periodic solution is a condition of dynamic equilibrium and, as such, may be stable or unstable. If the amplitude is perturbed slightly from the value for a periodic solution, the perturbation may grow or decay. If both positive and negative perturbations grow in absolute magnitude, the periodic solution is unstable while if both positive and negative perturbations decay, the periodic solution is stable.

From Equation (3[illegible]), we see that if r were nonzero, the values of n, τ_0, and s would be different from those found for a periodic solution. In particular, if r were of order ϵ^2, the modifications in n, τ_0, and s would be of second order. Since, the perturbations in n, τ_0, and s are of order ϵ^2, the stability analysis may be performed for solutions only in a range of order ϵ^2 near the neutral line. Therefore, we say

$$r = \epsilon^2 r^{(2)} \qquad (60)$$

It has been found that the solution to Equation (33) which results from separation of the first order terms in Equation (14) is of the form

$$Q_1(\beta)=\frac{1}{2}e^{r\beta}\left[e^{is\beta}+e^{-is\beta}\right] \tag{61}$$

Since $r=\epsilon^2 r^{(2)}$, the growth (or decay) of the first harmonic term is a third order effect. The effect of r on $\epsilon^2 R_2'$ and $\epsilon^3 R_3'$ is of fourth and fifth order, respectively, such that the aperiodicity of the second and third harmonics are negligible for our purposes.

It is convenient to redefine the transformation (35) in the following manner

$$\psi_{(1)\alpha}=(s-ir)\alpha \quad ; \quad \psi_{(2)\alpha}=(s+ir)\alpha$$
$$\psi_{(1)\beta}=(s-ir)\beta \quad ; \quad \psi_{(2)\beta}=(s+ir)\beta$$
$$\phi_{(1)}=\phi-irb_o \quad ; \quad \phi_{(2)}=\phi+irb_o$$

Also, let

$$\delta_{(1)}\equiv s-ir \qquad \delta_{(2)}\equiv s+ir$$

Note that from Equations (52) and (60), it is seen that

$$\phi_{(1)}=\phi^{(0)}+\epsilon^2\left(\phi^{(2)}-ir^{(2)}b_o^{(0)}\right)+O(\epsilon^3)$$
$$\phi_{(2)}=\phi^{(0)}+\epsilon^2\left(\phi^{(2)}+ir^{(2)}b_o^{(0)}\right)+O(\epsilon^3)$$
$$\delta_{(1)}=s^{(0)}+\epsilon^2\left(s^{(2)}-ir^{(2)}\right)+O(\epsilon^3)$$
$$\delta_{(2)}=s^{(0)}+\epsilon^2\left(s^{(2)}+ir^{(2)}\right)+O(\epsilon^3)$$

Consideration of Equation (61) and of the above statements leads to the conclusions that

$$Q_1(\beta) = \tfrac{1}{2}\left[e^{i\psi_{(1)\beta}} + e^{-i\psi_{(2)\beta}}\right]$$

$$Q_1(1+\beta) = \tfrac{1}{2}\left[e^{i(s^{(0)}+\psi_{(1)\beta})}\left\{1+i\epsilon^2(s^{(2)}-ir^{(2)})\right\} + e^{-i(s^{(0)}+\psi_{(2)\beta})}\left\{1-i\epsilon^2(s^{(2)}+ir^{(2)})\right\}\right]$$

$$Q_1(1+\beta-b_0) = \tfrac{1}{2}\left[e^{i(s^{(0)}+\psi_{(1)\beta}-\phi^{(0)})}\left\{1+i\epsilon^2(s^{(2)}-\phi^{(2)}-ir^{(2)}+ir^{(2)}b_0^{(0)})\right\} + \right.$$

$$\left. + e^{-i(s^{(0)}+\psi_{(2)\beta}-\phi^{(0)})}\left\{1-\epsilon^2(s^{(2)}-\phi^{(2)}+ir^{(2)}-ir^{(2)}b_0^{(0)})\right\}\right] + O(\epsilon^3)$$

$$Q_1(\beta-b_0) = \tfrac{1}{2}\left[e^{i(\psi_{(1)\beta}-\phi^{(0)})}\left\{1-i\epsilon^2(\phi^{(2)}-ir^{(2)}b_0^{(0)})\right\} + \right.$$

$$\left. + e^{-i(\psi_{(2)\beta}-\phi^{(0)})}\left\{1+i\epsilon^2(\phi^{(2)}+ir^{(2)}b_0^{(0)})\right\}\right] + O(\epsilon^3)$$

If these terms are substituted on the left-hand side of Equation (1.) the third order equation after separation becomes

$$K\left[1+u_0(1-\gamma n^{(0)})\right]Q_3(s^{(0)}+\psi_\beta) - \left[1-u_0(1-\gamma n^{(0)})\right]Q_3(\psi_\beta)$$

$$+\gamma n^{(0)}u_0\left[K\,Q_3(s^{(0)}+\psi_\beta-\phi^{(0)}) + Q_3(\psi_\beta-\phi^{(0)})\right] +$$

$$+n^{(2)}\left[(\omega_1+iy_1)e^{i\psi_\beta} + (\omega_1-iy_1)e^{-i\psi_\beta}\right] + \left[(s^{(2)}-ir^{(2)})(\omega_2+iy_2)e^{i\psi_\beta}\right.$$

$$\left.+(s^{(2)}+ir^{(2)})(\omega_2-iy_2)e^{-i\psi_\beta}\right] + \left[(\phi^{(2)}-ir^{(2)}b_0^{(0)})(\omega_3+iy_3)e^{i\psi_\beta} +\right.$$

$$\left.+(\phi^{(2)}+ir^{(2)}b_0^{(0)})(\omega_3-iy_3)e^{-i\psi_\beta}\right] = 2u_0R_3' =$$

$$= 2u_0\left[(b_r+ib_i)e^{3i\psi_\beta} + (b_r-ib_i)e^{-3i\psi_\beta} +\right.$$

$$\left.+(l_r+il_i)e^{i\psi_\beta} + (l_r-il_i)e^{-i\psi_\beta}\right] \qquad (62)$$

where $\omega_1, \omega_2, \omega_3, \psi_1, \psi_2, \psi_3, \ell_r$ and ℓ_i are defined in the previous section. Note that $\psi_{(1)\beta}$ and $\psi_{(2)\beta}$ are replaced by ψ_β since the error would not appear until fifth order. Furthermore, the series (42) has been used for n and the result for R_3' has been taken from the previous section. The first and second equations remain unchanged (after the above-mentioned substitution) from those equations found in the previous sections. In other words, up to and including third order terms, r has only a third order effect on the first harmonic.

As already mentioned, the first harmonic is contained solely in the C_1 term and, therefore not in the C_3 term. With this understanding, separation of the coefficients of $e^{i\psi_\beta}$ and $e^{-i\psi_\beta}$ in Equation (62) leads to the following

$$(\omega_1 + i\psi_1)n^{(2)} + (\omega_2 + i\psi_2)(s^{(2)} - ir^{(2)}) + (\omega_3 + i\psi_3)(\phi^{(2)} - ir^{(2)}b_0^{(0)}) = 2u_0(\ell_r + i\ell_i)$$
$$(\omega_1 - i\psi_1)n^{(2)} + (\omega_2 - i\psi_2)(s^{(2)} + ir^{(2)}) + (\omega_3 - i\psi_3)(\phi^{(2)} + ir^{(2)}b_0^{(0)}) = 2u_0(\ell_r - i\ell_i)$$

Separation of the real and imaginary parts of each of the above equations leads to the identical results:

$$\omega_1 n^{(2)} + \omega_2 s^{(2)} + \omega_3 \phi^{(2)} + (\psi_2 + \psi_3 b_0^{(0)})r^{(2)} = 2u_0 \ell_r$$
$$\psi_1 n^{(2)} + \psi_2 s^{(2)} + \psi_3 \phi^{(2)} - (\omega_2 + \omega_3 b_0^{(0)}) r^{(2)} = 2u_0 \ell_i$$

Taking note of the definitions following Equation (56), these above relations may be rewritten in the form

$$\begin{aligned} \omega_1 n^{(2)} + \omega_2' s^{(2)} + \omega_3' \tau_0^{(2)} + \psi_2' r^{(2)} &= 2u_0 \ell_r \\ \psi_1 n^{(2)} + \psi_2' s^{(2)} + \psi_3' \tau_0^{(2)} - \omega_2' r^{(2)} &= 2u_0 \ell_i \end{aligned} \qquad (63)$$

$s^{(2)}$ may be eliminated from the above system of equations and the following substitutions may be employed

$$n^{(2)} = \frac{1}{\epsilon^2} n' + O(\epsilon)$$
$$\tau_o^{(2)} = \frac{1}{\epsilon^2} \tau_o' + O(\epsilon)$$
$$r^{(2)} = \frac{1}{\epsilon^2} r + O(\epsilon)$$

with the final result:

$$[\omega_1 \psi_2' - \omega_2' \psi_1] n' + [\omega_3' \psi_2' - \psi_3' \omega_2'] \tau_o' + [(\omega_2')^2 + (\psi_2')^2] r = 2u_o \epsilon^2 [\ell_r \psi_2' - \omega_2' \ell_i] \quad (6\ )$$

If r were set equal to zero note the identity with Equations (57) and (58). Suppose ϵ^* were defined as that value of ϵ which satisfies Equation (58) for a given τ_o' and n' (i.e., the value of ϵ which gives a periodic solution). Then, it is seen that

$$[\omega_1 \psi_2' - \omega_2' \psi_1] n' + [\omega_3' \psi_2' - \psi_3' \omega_2'] \tau_o' = 2u_o \epsilon^{*2} [\ell_r \psi_2' - \omega_2' \ell_i] \quad (65)$$

Substitution into Equation (64) has the result

$$[(\omega_2')^2 + (\psi_2')^2] r = 2u_o (\epsilon^2 - \epsilon^{*2}) [\ell_r \psi_2' - \omega_2' \ell_i]$$

Note that the coefficient of r is always positive and, therefore, r has the same sign as $(\epsilon^2 - \epsilon^{*2})[\ell_r \psi_2' - \omega_2' \ell_i]$.

If $[\ell_r \psi_2' - \omega_2' \ell_i] > 0$, a positive perturbation $(\epsilon - \epsilon^*) > 0$ grows since $r > 0$ while a negative perturbation where $(\epsilon - \epsilon^*) < 0$ becomes more negative since $r < 0$. This means the periodic solution is unstable whenever $[\ell_r \psi_2' - \omega_2' \ell_i] > 0$. If this factor were less than zero, $(\epsilon - \epsilon^*) > 0$ means $r < 0$ and $(\epsilon - \epsilon^*) < 0$ means $r > 0$. So, the periodic solution is stable whenever $[\ell_r y_2' - \omega_2' \ell_i] < 0$. Note that $r = O(\epsilon^3)$ if $[\ell_r y_2' - \omega_2' \ell_i] = 0$.

It is most useful to investigate the effect of the sign of this factor upon the signs of n' and τ_0'. It is necessary to first determine the signs of the coefficients of n' and τ_0' in Equation (65); or, in other words, the signs of the factors $[\omega_1 y_2' - \omega_2' y_1]$ and $[\omega_3' y_2' - y_3' \omega_2']$ must be determined. This is readily accomplished if higher order terms in u_0 are neglected. Noting that $K = 1 + O(u_0)$ and $\delta^{(0)} = 2\ell\pi + O(u_0)$ in the definitions following Equations (52) and (56), we find that

$$\omega_1 \equiv \gamma u_0 \left[-1 + \cos\phi^{(0)}\right] + O(u_0^2)$$

$$y_1 \equiv -\gamma u_0 \sin\phi^{(0)} + O(u_0^2)$$

$$\omega_2 \equiv \frac{\gamma n^{(0)} u_0}{2} \sin\phi^{(0)} \mp \gamma n^{(0)} u_0 \left[1 - \left(1 - \frac{\gamma+1}{2\gamma n^{(0)}}\right)^2\right]^{1/2} + O(u_0^2)$$

$$y_2 \equiv \frac{K}{2} + \frac{u_0}{2} + \frac{\gamma n^{(0)} u_0}{2}\left(\cos\phi^{(0)} - 1\right) + O(u_0^2)$$

$$\omega_3 \equiv -\gamma n^{(0)} u_0 \sin\phi^{(0)} + O(u_0^2)$$

$$y_3 \equiv -\gamma n^{(0)} u_0 \cos\phi^{(0)} + O(u_0^2)$$

$$\omega_2' \equiv \gamma n^{(0)} u_0 \left\{\left[\tfrac{1}{2} - b_0^{(0)}\right]\sin\phi^{(0)} \mp \left[1 - \left(1 - \frac{\gamma+1}{2\gamma n^{(0)}}\right)^2\right]^{1/2}\right\} + O(u_0^2)$$

$$y_2' \equiv \frac{K}{2} + \frac{u_0}{2} + \gamma n^{(0)} u_0 \left[\left(\tfrac{1}{2} - b_0^{(0)}\right)\cos\phi^{(0)} - \tfrac{1}{2}\right] + O(u_0^2)$$

$$\omega_3' \equiv -4\ell\pi\,\gamma n^{(0)} u_0 \sin\phi^{(0)} + O(u_0^2)$$

$$y_3' \equiv -4\ell\pi\,\gamma n^{(0)} u_0 \cos\phi^{(0)} + O(u_0^2)$$

It is seen that y_2 and y_2' are each equal to $1/2 + O(u_0)$ while the remaining terms are all of order u_0. Therefore, neglecting contributions of order u_o^2, we find that

$$[\omega_1 \psi_2' - \omega_2' \psi_1] = \frac{\gamma u_o}{2}[-1+\cos\phi^{(o)}] + O(u_o^2)$$

$$[\omega_3' \psi_2' - \psi_3' \omega_2'] = -2\ell\pi\gamma n^{(o)} u_o \sin\phi^{(o)} + O(u_o^2)$$

The first term (which is the coefficient of n') is always negative while the second term (which is the coefficient of τ_o') is negative for $2m\pi < \phi^{(o)} < (2m+1)\pi$ and positive for $(2m+1)\pi < \phi^{(o)} < 2(m+1)\pi$.

Note that if $\phi^{(o)} = \omega^{(o)}\tau_o^{(o)} < (2m+1)\pi$, $n' < 0$ and (or) $\tau_o' < 0$ means a displacement outward (into the region of linear stability) while $n' > 0$ and (or) $\tau_o' > 0$ means an inward displacement (into the region of linear instability). If $\phi^{(o)} > (2m+1)\pi$, $n' < 0$ and (or) $\tau_o' > 0$ means an outward displacement while $n' > 0$ and (or) $\tau_o' < 0$ means an inward displacement.

Whenever $[\ell_r \psi_2' - \omega_2' \ell_i] > 0$, n' and τ_o' are shown by Equation (65) to be such that the displacement from the neutral line is outward into the region of linear stability (see Figure 10a). Whenever $[\ell_r \psi_2' - w_2' \ell_i] < 0$, n' and τ_o' are such that the displacement is inward into the region of linear instability (see Figure 10b). Of course, whenever $[\ell_r \psi_2' - \omega_2' \ell_i] = 0$, the displacement is tangent to the neutral line. Note that although this analysis involving the directions of the displacements assumes that u_o^2 is negligible, the conclusions are in agreement with numerical calculations (in which terms of order u_o^3 are implicitly neglected due to approximations in the solutions for $s^{(o)}$ and $\phi^{(o)}$).

It may be concluded that whenever $[\ell_r \psi_2' - \omega_2' \ell_i]$ is positive, unstable periodic solutions of amplitude ϵ are found at a distance $O(\epsilon^2)$ from the neutral line in the region where a small perturbation decays (linear stability). This is shown schematically in Figure 10c which is a cross-sectional plot of the three-dimensional τ_o vs. n vs. ϵ plot. ϵ is plotted as the ordinate while a line normal to the $\tau_o^{(o)}$ vs. $n^{(o)}$ neutral line and lying in the $\epsilon = 0$ plane is plotted as the abcissa. The neutral line is supposed to be passing through the origin such that ϵ vs. normal displacement from the neutral line is plotted. An outward displacement is taken to the right and an inward displacement is taken to the left. The curve is parabolic, of course, and gives the locus of points where periodic solutions are found. Any solution to the left of (and above) this curve grows in amplitude with time while any solution to the right of (and below) decays in amplitude with time.

This indicates the possibility of "triggering" action since disturbances of certain amplitudes or greater grow while others decay. Although a small disturbance may not grow into a finite size oscillation, a finite disturbance may result in unstable engine operation.

While it has been indicated that disturbances above a certain amplitude grow, there has been no indication from this analysis as to the final regime condition reached. That is, no stable periodic solution of higher amplitude has been found. This is most likely explained (but not proven here) by the fact that the stable periodic solution contains shock waves which were excluded in this analysis. In other words, a regime condition similar to that found for the case studied in Chapter II may be expected.

Whenever $[\ell_r \gamma_2' - \omega_2' \ell_i]$ is negative, it may be concluded that the stable periodic solutions of amplitude ϵ are found at a distance $O(\epsilon^2)$ from the neutral line in a region the region where a small perturbation grows (linear instability). Figure 10d shows the parabola which is the locus of points where a periodic solution exists. Any solution to the left of (and below) this curve grows in amplitude until the amplitude of the periodic solution is reached while any solution to the right of (and above) this curve decays. If the displacement is inward, the amplitude decays until the amplitude of the periodic solution is obtained while if the displacement is outward the amplitude of any disturbance decays to zero. In this case, a periodic solution without shock waves has been found.

Note that the conclusions shown in Figures 10c and 10d are in accordance with those found by Crocco who, in effect, said that for $\epsilon = 0$, these small disturbances grow for inward displacements (to the left) and decay for outward displacements (to the right).

One may wonder about the significance of this stability analysis being performed in a relatively simple manner. It is well-known that the stability analysis of a periodic solution satisfying an ordinary differential equation is usually a difficult task to perform (Ref. 18). The stability of the periodic solutions found for the ordinary differential equation of Appendix A is probably not obtainable, for instance. Here, however, periodic solutions which satisfy partial differential equations have been found and their stability has been analyzed without much difficulty. The reason for the simplicity of this analysis is that the stability criterion is related to the boundary conditions and not to the partial differential equations. Under the assumptions, all energy addition or removal occurs at the boundaries (combustion zone and nozzle) and none occurs in the flow field (chamber). The stability analysis is really performed on periodic solutions which satisfy boundary conditions stated as algebraic relations and the stability analysis is not performed on the solution to the partial differential equations. This is clearly seen by the fact that the solutions to the differential equations are the Riemann invariants which do not grow or decay in magnitude but, of course, remain invariant. So, the stability analysis has been performed for periodic solutions which essentially satisfy algebraic equations. This type of analysis, therefore, might be expected to be somewhat simpler than that performed for the solution to an ordinary differential equation such as that presented in Appendix A.

As a side point of interest, it is noted that an order of magnitude argument used with Equations (63) shows that $s^{(2)}$ and $r^{(2)}$ are each of order u_o. This argument assumes that $\tau_o^{(2)}$, $n^{(2)}$, ℓ_r, and ℓ_i are each of order unity or smaller.

WAVE FORM OF THE OSCILLATIONS

The wave forms of the stable periodic solutions would be useful to determine. In particular, pressure versus time graphs at fixed space positions are of interest since these are most readily found experimentally.

Pressure is determined directly from the speed of sound by means of the isentropic relationship which combined with the series expansion (2) yields the result

$$(1+\epsilon p_1 + \epsilon^2 p_2 + \epsilon^3 p_3) = (1+\epsilon a_1 + \epsilon^2 a_2 + \epsilon^3 a_3)^{\frac{2\gamma}{\gamma-1}} + O(\epsilon^4)$$

A binomial expansion of the right-hand side and separation according to powers in ϵ yields the following

$$p_1 = \frac{2\gamma}{\gamma-1} a_1$$

$$p_2 = \frac{2\gamma}{\gamma-1} a_2 + \frac{\gamma(\gamma+1)}{(\gamma-1)^2} a_1^2$$

$$p_3 = \frac{2\gamma}{\gamma-1} a_3 + \frac{2\gamma(\gamma+1)}{(\gamma-1)^2} a_1 a_2 + \frac{2\gamma(\gamma+1)}{3(\gamma-1)^3} a_1^3 \tag{66}$$

Now, p, x, and t have been determined as functions of the coordinates α and β. A general relationship between α and β representing a curve which when transformed back to x, t coordinates is a straight line at constant x would be desirable. This is easily found only for $x = 0$ (combustion zone) where $\alpha = \beta$ and for $x = 1$ (nozzle entrance) where $\alpha = 1 + \beta$. At each of these locations $\tau_1 = \tau_2 = 0$ so that the transformation is simplified. The pressure vs. time wave forms will be plotted at each of these two locations.

In order to plot the wave form at $x = 0$, we must calculate $p_i(\psi_\beta, \psi_\beta)$ and $t_i(\psi_\beta, \psi_\beta)$. Combination of Equations (39), (48), (55), and (66) yields the results for p_i

$$p_1 = \frac{\gamma}{2}\left[K \cos(s + \psi_\beta) + \cos \psi_\beta\right]$$

$$\begin{aligned}
p_2 = {} & \gamma A\left[\cos 2(\psi_\beta + \theta) + K \cos 2(\psi_\beta + s + \theta)\right] \\
& + \frac{\gamma}{2}(1+K)C_r + \frac{\gamma(\gamma+1)}{16}\left[\frac{K^2+1}{2} + K \cos s\right. \\
& \left. + \frac{K^2}{2}\cos 2(s+\psi_\beta) + (K+\tfrac{1}{2})\cos 2\psi_\beta - K \sin s \sin 2\psi_\beta\right]
\end{aligned}$$

$$\begin{aligned}
p_3 = {} & \gamma B\left[K \cos 3(\psi_\beta + s + \Delta) + \cos 3(\psi_\beta + \Delta)\right] \\
& + \frac{\gamma(\gamma+1)}{8}\Big\{A\Big[\cos(3\psi_\beta + 2\theta) + \cos(\psi_\beta + 2\theta) + \\
& + K\big(\cos(3\psi_\beta + 2\theta + s) + \cos(\psi_\beta + 2\theta - s) \\
& + \cos(3\psi_\beta + 2s + 2\theta) + \cos(\psi_\beta + 2s + 2\theta)\big) + \\
& + K^2\big(\cos(3\psi_\beta + 3s + 2\theta) + \cos(\psi_\beta + s + 2\theta)\big)\Big] + \\
& + (1+K)C_r\left[K \cos(s + \psi_\beta) + \cos \psi_\beta\right]\Big\} +
\end{aligned}$$

$$+\frac{\gamma(\gamma+1)}{96}\left\{\frac{K^3}{4}\left[\cos 3(s+\psi_\beta)+3\cos(s+\psi_\beta)\right]+\right.$$

$$+\frac{3K^2}{4}\left[2\cos\psi_\beta+\cos(3\psi_\beta+2s)+\cos(\psi+2s)\right]$$

$$+\frac{3K}{4}\left[2\cos(s+\psi_\beta)+\cos(3\psi_\beta+s)+\cos(\psi_\beta-s)\right]$$

$$\left.+\frac{1}{4}\left[\cos 3\psi_\beta+3\cos\psi_\beta\right]\right\} \qquad (67)$$

It is seen that there are corrections of the order ϵ^2 to the average pressure and only part of this correction is proportional to the factor C_r.

Equations (10), (35), (41) and (50) yields the results for $t_i(\psi_\beta, \psi_\beta)$ as follows

$$t_0 = \frac{2\psi_\beta}{s(1-u_0^2)}$$

$$t_1 = -\frac{\gamma+1}{8}\,\frac{\left[K\left(\frac{1}{1+u_0}\right)^2+\left(\frac{1}{1-u_0}\right)^2\right]}{1-\cos s}\left[\cos\psi_\beta+\cos s-\right.$$

$$\left.-\cos(s+\psi_\beta)-1\right]+\frac{3-\gamma}{4s}\left\{K\left(\frac{1}{1-u_0}\right)^2\left[\sin(s+\psi_\beta)-\sin s\right]\right.$$

$$\left.+\left(\frac{1}{1+u_0}\right)^2\sin\psi_\beta\right\}$$

$$t_2 = \frac{\psi_\beta}{s}\left\{V_3+\frac{3-\gamma}{4}\left[\left(\frac{1}{1+u_0}\right)^2+\frac{K}{(1-u_0)^2}\right]C_r+\right.$$

$$+\frac{3-\gamma}{16}\left(\frac{1-u_0}{1+u_0}\,s\,\frac{\gamma+1}{8}\left[\frac{K}{(1+u_0)^2}+\frac{1}{(1-u_0)^2}\right]\frac{\sin s}{1-\cos s}+\right.$$

$$\left.+\frac{1+u_0}{1-u_0}\,Ks\,\frac{\gamma+1}{8}\left[\frac{K}{(1+u_0)^2}+\frac{1}{(1-u_0)^2}\right]\frac{\sin 2s-\sin s}{1-\cos s}\right)$$

$$+\frac{3-\gamma}{32}\left(\left[\frac{\gamma+1}{4(1-u_0^2)}+\frac{3-\gamma}{(1-u_0)^3}\right]K^2+\left[\frac{\gamma+1}{4(1-u_0^2)}+\frac{3-\gamma}{(1+u_0)^3}\right]\right)\Bigg\}+$$

$$+\frac{3-\gamma}{4}\left\{\left(\frac{1}{1+u_0}\right)^2\frac{A}{s}\sin 2(\psi_\beta+\theta)+\frac{AK\sin 2(\psi_\beta+s+\theta)}{s(1-u_0)^2}\right.$$

$$-\left(\frac{1}{1+u_0}\right)^2\frac{A}{s}\sin 2\theta-\left(\frac{1}{1-u_0}\right)^2\frac{AK}{s}\sin 2(\theta+s)$$

$$-\frac{\gamma+1}{64}\left[\frac{K}{(1+u_0)^2}+\frac{1}{(1-u_0)^2}\right]\left(\frac{1-u_0}{1+u_0}\left[1+\frac{\cos\psi_\beta-\cos(\psi_\beta-s)}{1-\cos s}\right]\right.$$

$$\left.-K\,\frac{1+u_0}{1-u_0}\left[1-\frac{\cos\psi_\beta-\cos(\psi_\beta+s)}{1-\cos s}\right]\right)+$$

$$+\frac{K^2}{16s}\left[\frac{\gamma+1}{4(1-u_0^2)}+\frac{3-\gamma}{(1-u_0)^3}\right]\left(\sin 2(\psi_\beta+s)-\sin 2s\right)$$

$$\left.+\frac{1}{16s}\left[\frac{\gamma+1}{4(1-u_0^2)}+\frac{3-\gamma}{(1-u_0)^3}\right]\left(\sin 2\psi_\beta-1\right)\right\}+$$

$$+\frac{1}{1-\cos 2s}\left\{r_1A\left(\cos 2(\psi_\beta+\theta)-\cos 2(\psi_\beta+\theta+s)\right.\right.$$

$$\left.-\cos 2\theta+\cos 2(s+\theta)\right)+\frac{r_2}{4}\left(\cos 2\psi_\beta-\cos 2(\psi_\beta+s)\right.$$

$$\left.-1+\cos 2s\right)+\frac{r_3K}{4s}\left(\sin(2\psi_\beta+s)-\sin 2\psi_\beta-\right.$$

$$\left.-\sin(2\psi_\beta+3s)+\sin 2(\psi_\beta+s)-\sin s+\sin 3s-\sin 2s\right)$$

$$+\frac{r_4}{4s}\left(\sin 2\psi_\beta-\sin(2\psi_\beta-s)-\sin 2(\psi_\beta+s)+\sin(2\psi_\beta+s)\right.$$

$$\left.\left.+\sin 2s-2\sin s\right)\right\} \qquad (68)$$

A plot of $\left(1+\epsilon p_1(\psi_\beta,\psi_\beta)\right)$ vs. $t_0(\psi_\beta,\psi_\beta)$ gives a first order approximation to the wave form at the combustion zone. Since the zero order pressure term is constant with time, it is not necessary to include the first order correction ϵt_1 for time. The difference between evaluating ϵp_1 at t_0 and at $(t_0+\epsilon t_1)$ is of order ϵ^2 and, therefore, negligible for the purpose of a first order approximation. A second order approximation to the wave form at the combustion zone is given by a plot of $\left(1+\epsilon p_1(\psi_\beta,\psi_\beta)+\epsilon^2 p_2(\psi_\beta,\psi_\beta)\right)$ vs. $\left(t_0(\psi_\beta,\psi_\beta)+\epsilon t_1(\psi_\beta,\psi_\beta)\right)$ while a third order approximation is given by a plot of

$$\left(1+\epsilon p_1(\psi_\beta,\psi_\beta)+\epsilon^2 p_2(\psi_\beta,\psi_\beta)+\epsilon^3 p_3(\psi_\beta,\psi_\beta)\right) \text{ vs. } \left(t_0(\psi_\beta,\psi_\beta)+\epsilon t_1(\psi_\beta,\psi_\beta)+\epsilon^2 t_2(\psi_\beta,\psi_\beta)\right).$$

The pressure wave form at the nozzle entrance is calculated in a similar fashion. Equations (39), (48), (55), and (66) are used to evaluate $p_i(\psi_\beta+s,\psi_\beta)$ with the following results:

$$p_1=\frac{\gamma}{2}(K+1)\cos(s+\psi_\beta)$$

$$p_2=\gamma(K+1)A\cos 2(\psi_\beta+s+\theta)+$$
$$+\frac{\gamma(\gamma+1)(K+1)^2}{32}\left[\cos 2(\psi_\beta+s)+1\right]+\frac{\gamma}{2}(K+1)C_r$$

$$p_3=\gamma(K+1)B\cos 3(\psi_\beta+s+\Delta)+$$
$$+\frac{\gamma(\gamma+1)}{8}(K+1)^2\Big\{A\left[\cos(3\psi_\beta+3s+2\theta)+\right.$$
$$\left.+\cos(\psi_\beta+s+2\theta)\right]+C_r\cos(\psi_\beta+s)\Big\}+$$
$$+\frac{\gamma(\gamma+1)(K+1)^3}{384}\left[\cos 3(\psi_\beta+s)+3\cos(\psi_\beta+s)\right] \qquad (69)$$

Equations (10), (35), (41), and (50) yield the following results for $t_i(\psi_\beta + s, \psi_\beta)$

$$t_o = \frac{1}{1+u_o} + \frac{2}{s}\frac{\psi_\beta}{1-u_o^2}$$

$$t_1 = \frac{\gamma+1}{8}\left[\frac{1}{1-u_o} - K\frac{1}{1+u_o}\right]\cos(s+\psi_\beta)$$

$$-\frac{\gamma+1}{16}\,\frac{\left[K\frac{1-u_o}{1+u_o} + \frac{1+u_o}{1-u_o}\right]}{(1-\cos s)}\left\{\frac{\cos(s+\psi_\beta) + \cos s - \cos(\psi_\beta + 2s) - 1}{1+u_o} +\right.$$

$$\left. + \frac{\cos\psi_\beta + \cos s - \cos(s+\psi_\beta) - 1}{1-u_o}\right\}$$

$$+ \frac{3-\gamma}{4s}\left\{\left[K\left(\frac{1}{1-u_o}\right)^2 + \left(\frac{1}{1+u_o}\right)^2\right]\sin(s+\psi_\beta) - \frac{K\sin s}{(1-u_o)^2}\right\}$$

$$t_2 = \frac{1-u_o}{2}V_3 + \frac{\gamma+1}{8}\left[\frac{1}{1-u_o} - \frac{K}{1+u_o}\right]C_r + \frac{3-\gamma}{4}\frac{C_r}{(1+u_o)^2}$$

$$+\left(\frac{\gamma+1}{16}\right)^2\left\{\left[(1-u_o)K - (1+u_o)\right]\left[\frac{K}{(1+u_o)^2} + \frac{1}{(1-u_o)^2}\right]\left[\frac{1-2\cos s + \cos 2s}{1-\cos s}\right]\right\}$$

$$+ \frac{3-\gamma}{16}\,\frac{\gamma+1}{8}\,\frac{1-u_o}{1+u_o}\,s\left[\frac{K}{(1+u_o)^2} + \frac{1}{(1-u_o)^2}\right]\frac{\sin s}{1-\cos s} -$$

$$- \frac{(\gamma+1)(3-\gamma)}{4}\,\frac{K}{s}\left[\frac{3+u_o}{(1+u_o)^2} - \frac{3-u_o}{(1-u_o)^2}\right]\sin s + \frac{3-\gamma}{32}\left[\frac{\gamma+1}{4(1-u_o^2)} + \frac{3-\gamma}{(1+u_o)^3}\right]$$

$$+ \frac{\gamma+1}{32}\left\{-\frac{(\gamma+1)u_oK}{2(1-u_o^2)} + \frac{1}{2}\left[\frac{3-\gamma}{4}\frac{1}{1-u_o} - \frac{\gamma+1}{2}\frac{1}{1+u_o}\right] + \frac{3\gamma-1}{8}\frac{1}{1+u_o}\right\}$$

$$+\left\{V_3 + \frac{3-\gamma}{4}\left[\frac{1}{(1+u_o)^2} + \frac{K}{(1-u_o)^2}\right]C_r +\right.$$

$$+ \frac{3-\gamma}{16}\,\frac{1-u_o}{1+u_o}\,\frac{\gamma+1}{8}\,s\left[\frac{K}{(1+u_o)^2} + \frac{1}{(1-u_o)^2}\right]\frac{\sin s}{1-\cos s} +$$

$$+ \frac{1+u_0}{1-u_0} K s \; \frac{\gamma+1}{8} \left[\frac{K}{(1+u_0)^2} + \frac{1}{(1-u_0)^2} \right] \frac{\sin 2s - \sin s}{1 - \cos s}$$

$$+ \frac{3-\gamma}{32} \left[\frac{\gamma+1}{4(1-u_0^2)} + \frac{3-\gamma}{(1-u_0)^3} \right] K^2 + \frac{3-\gamma}{32} \left[\frac{\gamma+1}{4(1-u_0^2)} + \frac{3-\gamma}{(1+u_0)^3} \right] \Bigg\} \frac{\psi_\beta}{s}$$

$$+ \frac{r_1 A}{1-\cos 2s} \left\{ \frac{1-u_0}{2} \left[\cos 2(\theta + s + \psi_\beta) - \cos 2(\theta + 2s + \psi_\beta) \right] \right.$$

$$\left. + \frac{1+u_0}{2} \left[\cos 2(\psi_\beta + \theta) - \cos 2(\psi_\beta + \theta + s) \right] \right\} +$$

$$+ \frac{r_2}{4(1-\cos 2s)} \left\{ \frac{1-u_0}{2} \left[\cos 2(\psi_\beta + s) - \cos 2\psi_\beta \right] + \right.$$

$$\left. + \frac{1+u_0}{2} \left[\cos 2\psi_\beta - \cos 2(\psi_\beta - s) \right] \right\} + \frac{r_3 K}{4s(1-\cos 2s)} \left\{ \frac{1-u_0}{2} \left[\sin(3s + 2\psi_\beta) \right. \right.$$

$$\left. - \sin 2(\psi_\beta + s) - \sin(2\psi_\beta + 5s) + \sin 2(\psi_\beta + 2s) \right] +$$

$$\left. + \frac{1+u_0}{2} \left[\sin(2\psi_\beta + s) - \sin 2\psi_\beta - \sin(2\psi_\beta + 3s) + \sin 2(\psi_\beta + s) \right] \right\}$$

$$+ \frac{r_4}{4s(1-\cos 2s)} \left\{ \frac{1-u_0}{2} \left[\sin 2(\psi_\beta + s) - \sin(2\psi_\beta + s) - \sin 2(\psi_\beta + 2s) \right. \right.$$

$$\left. + \sin(2\psi_\beta + 3s) \right] + \frac{1+u_0}{2} \left[\sin 2\psi_\beta - \sin(2\psi_\beta - s) - \right.$$

$$\left. \left. - \sin 2(\psi_\beta + s) + \sin(2\psi_\beta + s) \right] \right\} - \frac{r_1 A \left[\cos 2\theta - \cos 2(s+\theta) \right]}{1 - \cos 2s}$$

$$- \frac{r_2}{4} - \frac{r_3 K \left[\sin s - \sin 3s + \sin 2s \right]}{4s(1-\cos 2s)} - \frac{r_4 \left[2 \sin s - \sin 2s \right]}{4s(1-\cos 2s)}$$

$$+ \frac{\gamma+1}{4} A \left[\frac{1}{1-u_0} - \frac{K}{1+u_0} \right] \cos 2(\psi_\beta + \theta + s) + \frac{3-\gamma}{4s} A \left\{ \frac{K}{(1-u_0)^2} \sin 2(\theta + s) \right.$$

$$\left. - \frac{1}{(1+u_0)^2} \sin 2\theta + \left[\frac{1}{(1+u_0)^2} + \frac{K}{(1-u_0)^2} \right] \sin 2(\theta + s + \psi_\beta) \right\}$$

$$+\left(\frac{\gamma+1}{16}\right)^2 (1+u_o)\left[\frac{K}{(1+u_o)^2}+\frac{1}{(1-u_o)^2}\right]\frac{\cos(2\psi_\beta+s)-\cos 2(\psi_\beta+s)}{1-\cos s}+$$

$$+\left(\frac{\gamma+1}{16}\right)^2\left[(1-u_o)K-(1+u_o)\right]\left[\frac{K}{(1+u_o)^2}+\frac{1}{(1-u_o)^2}\right]\frac{\cos 2\psi_\beta-\cos(2\psi_\beta+s)}{1-\cos s}$$

$$+\frac{\gamma+1}{8}\left(\frac{1-u_o}{32}\right)\left[\frac{3-\gamma}{1+u_o}-(\gamma+1)K\right]\left[\frac{K}{(1+u_o)^2}+\frac{1}{(1-u_o)^2}\right]\frac{\cos(2\psi_\beta+s)-\cos 2(\psi_\beta+s)}{1-\cos s}$$

$$+\frac{3-\gamma}{16}\ \frac{\gamma+1}{16}\left[\frac{K}{(1+u_o)^2}+\frac{1}{(1-u_o)^2}\right]\left\{\frac{1-u_o}{1+u_o}+K\frac{1+u_o}{1-u_o}\left[\cos 2\psi_\beta-1\right.\right.$$

$$\left.\left.+\cos s-\cos(2\psi_\beta+s)\right]\frac{1}{1-\cos s}\right\}+$$

$$+\frac{(\gamma+1)(3-\gamma)(3-u_o)K}{4(1-u_o)^2\, s}\left[\sin(2\psi_\beta+3s)-\sin 2(\psi_\beta+s)\right]$$

$$+\frac{(\gamma+1)(3-\gamma)(3+u_o)K}{4(1+u_o)^2\, s}\left[\sin 2(\psi_\beta+s)-\sin(2\psi_\beta+s)\right]$$

$$+\frac{3-\gamma}{16}\left[\frac{\gamma+1}{4(1-u_o^2)}+\frac{3-\gamma}{(1-u_o)^3}\right]\frac{K^2}{4s}\left[\sin 2(\psi_\beta+s)-\sin 2s\right]$$

$$+\frac{3-\gamma}{16}\left[\frac{\gamma+1}{4(1-u_o^2)}+\frac{3-\gamma}{(1+u_o)^3}\right]\frac{\sin 2(\psi_\beta+s)}{4s}+$$

$$+\frac{\gamma+1}{32}\left\{\frac{3-\gamma}{8}\ \frac{1}{1-u_o}-\frac{\gamma+1}{4}\left[\frac{1}{1+u_o}+\frac{2Ku_o}{1-u_o^2}\right]+\frac{3\gamma-1}{8}\ \frac{1}{1+u_o}\right\}\cos 2(\psi_\beta+s) \tag{70}$$

A first order approximation to the pressure wave form at the nozzle entrance is given by a plot of $(1+\epsilon p_1(s+\psi/\beta,\psi/\beta))$ vs. $t_0(s+\psi/\beta,\psi/\beta)$. A second order approximation is given by a plot of $\left(1+\epsilon p_1(s+\psi/\beta,\psi/\beta)+\epsilon^2 p_2(s+\psi/\beta,\psi/\beta)\right)$ vs. $\left(t_0(s+\psi/\beta,\psi/\beta)+\epsilon t_1(s+\psi/\beta,\psi/\beta)\right)$ while a third order approximation is given by a plot of $\left(1+\epsilon p_1(s+\psi/\beta,\psi/\beta)+\epsilon^2 p_2(s+\psi/\beta,\psi/\beta)+\epsilon^3 p_3(s+\psi/\beta,\psi/\beta)\right)$ vs. $\left(t_0(s+\psi/\beta,\psi/\beta)+\epsilon t_1(s+\psi/\beta,\psi/\beta)+\epsilon^2 t_2(s+\psi/\beta,\psi/\beta)\right)$.

NUMERICAL EXAMPLE:

In any calculation based on these results it is necessary to specify the rate function $f(a)$. This is done by stating the values of the coefficients in a Taylor series expansion about the steady-state value. For an approximation with error of fourth order in the speed of sound perturbation, only the first three coefficients $N\left(\equiv\frac{2\gamma}{\gamma-1}n\right)$, M, and L, need be stated (See Appendix E). These coefficients are readily calculated for any rate function which is analytic.

For the purpose of numerical example, the function is chosen to be

$$f=p^{\mu}$$

where μ is some constant. The isentropic relationship says that

$$f=a^{\frac{2\gamma\mu}{\gamma-1}}$$

Therefore, it is found that

$$N \equiv \frac{1}{f}\frac{df}{da}\bigg|_{a=1} = \frac{2\gamma\mu}{\gamma-1}$$

$$M \equiv \frac{1}{2!}\frac{1}{f}\frac{d^2f}{da^2}\bigg|_{a=1} = \frac{\gamma\mu}{\gamma-1}\left(\frac{2\gamma\mu}{\gamma-1}-1\right)$$

$$L \equiv \frac{1}{3!}\frac{1}{f}\frac{d^3f}{da^3}\bigg|_{a=1} = \frac{2}{3}\frac{\gamma\mu}{\gamma-1}\left(\frac{\gamma\mu}{\gamma-1}-1\right)\left(\frac{2\gamma\mu}{\gamma-1}-1\right)$$

Since $n \equiv \frac{\gamma-1}{2\gamma} N$, it is immediately seen that n and μ are identical. n is the interaction index as defined by Crocco (Ref. 1).

According to experimental results (Ref. 2), the range of practical interest is $n < 2$. The minimum value of $n^{(o)}$ possible for unstable operation was found by Crocco to be $\frac{\gamma+1}{4\gamma}$. If $\gamma=1.2$, this minimum value is $n^{(o)}=.46$.

In the calculations, $\gamma=1.2$ was taken for all cases. Three cases were examined for the mean flow Mach number: $u=.1$, .2, and .3. Furthermore, three modes of oscillation were examined: the fundamental mode ($\ell=1$), the second harmonic mode ($\ell=2$), and the third harmonic mode ($\ell=3$). The integer m was taken equal to zero in all calculations.

$s^{(o)}$, $\phi^{(o)}$, and $\tau_o^{(o)}$ were calculated for various $n^{(o)}$ in the range $0.5 \leq n^{(o)} \leq 10.0$ by means of Equations (37) and (38). The results are double-valued since there is a choice of a plus or minus sign in these equations. Figures 11, 12, and 13 show the results for the case $\ell=1$ (fundamental mode) and $u=.2$. If the integer m were taken as nonzero in these calculations, there would be no change in the $s^{(o)}$ vs. $n^{(o)}$ curve (Fig. 11), the $\phi^{(o)}$ vs. $n^{(o)}$ curve (Fig. 12) would be translated vertically with no distortion, and the $\tau^{(o)}$ vs. $n^{(o)}$ curve (Fig. 13) would be translated vertically and distorted somewhat. The minimum value of $n^{(o)}$ would remain unchanged and the shapes of the curves would still be "parabolic-like."

At a value of n greater than the minimum value neutral oscillation is possible at two frequencies: one frequency greater than the natural resonant frequency ($s > 2\pi$) and the other frequency less than the natural resonant frequency ($s < 2\pi$). The frequency of oscillation is dependent upon the characteristic time of the combustion process τ_o (or ϕ). If the time of combustion increases so does the period of the oscillation and, therefore, the frequency decreases. If $\tau_o^{(o)} > 1$ $(\phi^{(o)} > \pi)$, we find $s^{(o)} < 2\pi$ while if $\tau_o^{(o)} < 1$ $(\phi^{(o)} < \pi)$, we have $s^{(o)} > 2\pi$. The two different branches of the curves of $s^{(o)}$ vs. $n^{(o)}$ and $\phi^{(o)}$ vs. $n^{(o)}$ are marked in Figures 11 and 12 as either $\tau_o^{(o)} > 1$ or $\tau_o^{(o)} < 1$ in order to distinguish between them.

The parameters A, θ and C_p which appear in the second order contribution to the wave form may be calculated as indicated by Equations (47a) and (47b). Note that there is no change in the final results if the sign of $A(n^{(o)})$ is changed provided that $\pm \frac{\pi}{2}$ is added to $\theta(n^{(o)})$. These three parameters depend upon $n^{(o)}$, $s^{(o)}(n^{(o)})$ and $\phi^{(o)}(n^{(o)})$.

Figures 14, 15, and 16 show the case $u = .2$ and $l = 1$ (fundamental mode). Figure 14 shows that A tends to plus infinity as $n^{(0)}$ tends towards $\frac{\gamma+1}{4\gamma}$. For $\tau_0^{(0)} < 1$, $A(n^{(0)})$ remains positive while for $\tau_0^{(0)} > 1$, $A(n^{(0)})$ becomes negative for higher values of $n^{(0)}$. $A(n^{(0)})$ is of order unity for a wide range, $0.6 < n^{(0)} < 10.0$ and above. Therefore, since the coefficient of the second harmonic term in the wave form is proportional to $\epsilon^2 A$, this coefficient is of order ϵ^2 . (Note that this applies to the wave form in characteristic coordinates and there are additional second harmonic contributions resulting from the transformation to space and time coordinates).

The phase term θ is plotted verses n in Figure 15 and is seen to change rapidly as $n^{(0)} \longrightarrow \frac{\gamma+1}{4\gamma}$. The wild behavior of θ for $\tau_0^{(0)} > 1$ occurs where $A = 0$ (see Figure 14) so that it does not result in any wild behavior in the wave form calculations. For larger values of $n^{(0)}$, θ increases for $\tau_0^{(0)} < 1$ and decreases for $\tau_0^{(0)} > 1$.

The factor C_r which is the coefficient of the correction to the mean flow condition becomes infinite as $n^{(0)}$ shrinks to its minimum value. C_r is negative for $\tau_0^{(0)} > 1$ and positive for $\tau_0^{(0)} < 1$. Its absolute value decreases as $n^{(0)}$ becomes larger in both cases.

The B and Δ terms are calculated by means of Equations (54). These are parameters in the third order contribution to the wave form; in particular, $\epsilon^3 B$ is the coefficient of the third harmonic term in the wave form and 3Δ is the phase. Note that there is no change in the wave form if the sign of B is changed provided that $\pm\frac{\pi}{3}$ is added to Δ . These two parameters depend upon $n^{(o)}$, $s^{(o)}$, $\phi^{(o)}$, A, θ, and C_r .

Figures 17 and 18 show the curves of B vs. $n^{(o)}$ and Δ vs. $n^{(o)}$ for the case $\varkappa = .2$ and $\ell = 1$ (fundamental mode). It is seen from Figure 17 that B becomes infinite as $n^{(o)} \rightarrow \frac{\gamma+1}{4\gamma}$. This parameter is negative for $\tau_o^{(o)} > 1$ and positive for $\tau_o^{(o)} < 1$. Note that B is usually somewhat larger than A such that $\epsilon^2 A$ and $\epsilon^3 B$ could become comparable if ϵ were not too small. This would mean that the second and third harmonic components in the wave form would be comparable in amplitude (in characteristic coordinates).

The Δ term becomes large as $n^{(o)} \rightarrow \frac{\gamma+1}{4\gamma}$. For $\tau_o^{(o)} > 1$, Δ decreases monotonically as $n^{(o)}$ increases and eventually becomes negative for larger $n^{(o)}$ while for $\tau_o^{(o)} < 1$, Δ increases monotionically with $n^{(o)}$ and becomes positive for larger $n^{(o)}$.

The factor D is calculated by means of Equation (59) and is seen to depend upon $n^{(o)}$, $\phi^{(o)}$, $s^{(o)}$, A , θ , and C_r . D is the factor in the relationships between displacement from the neutral line in a τ vs. n plot and the amplitude parameter ϵ . A positive D indicates an outward displacement and a negative D indicates an inward displacement.

The result for the $\varkappa = .2$ and $\ell = 1$ case is shown in Figure 19. For both $\tau_0^{(0)} > 1$ and $\tau_0^{(0)} < 1$, D becomes infinite as $n^{(0)} \longrightarrow \frac{\gamma+1}{4\gamma}$ and decreases as $n^{(0)}$ increases from this minimum value and eventually becomes negative. (For $\tau_0^{(0)} > 1$, it becomes positive again at still larger $n^{(0)}$). The change in sign occurs at approximately $n^{(0)}$ 2.25 for $\tau_0^{(0)} > 1$ and $n^{(0)} = 2.75$ for $\tau_0^{(0)} < 1$.

The results for all other combinations of u and ℓ are qualitatively similar; i.e., they may be described in the same manner that the $\ell = 1$, $u = .2$ case has been described here.

Note that positive D implies unstable periodic solutions while negative D implies stable periodic solutions. In the range of values of n of practical interest ($n < 2.0$ according to Ref. 2), only unstable solutions were found for all cases studied.

The determination of the wave form would only be of interest for stable solutions ($D \leq 0$). Wave form calculations are presented herein for the case: $\gamma = 1.2$, $u = 0.2$, $\epsilon = 0.1$, $\ell = 1$, and $n^{(0)} = 4.0$. At this value of $n^{(0)}$, the periodic solution is stable for both points $\tau_0^{(0)} > 1$ and $\tau_0^{(0)} < 1$. A fixed value of ϵ was chosen such that displacement is not the same for all points. Figures 20 and 21 show the wave forms at the nozzle while Figures 22 and 23 apply at the injector. Each figure shows three approximations to the wave form as explained in the previous section: first order, second order, and third order approximations. The wave form over the time period of oscillation is indicated.

Comparison of Figure 20 with Figure 21 (or Figure 22 with 23) shows significant difference between the wave form for $\tau_0^{(0)} > 1$ (below-resonant frequency) and $\tau_0^{(0)} < 1$ (above-resonant frequency). For above-resonant oscillations, the mean-pressure correction is positive and the positive peak becomes sharper. On the other hand, for below-resonant oscillations, the mean-pressure correction is negative and the negative peak becomes sharper.

A very important factor in the wave form is the co-ordinate correction which is related to the nonlinearity of the characteristic curves. It is seen that the correction ϵt_1, is governed by a nonhomogeneous wave equation with a periodic forcing function. The harmonic oscillator anology informs us that a phase shift of 180° occurs whenever the forcing function is changed from an above-resonant frequency to a below-resonant frequency. The result in our wave phenomenon is not this simple, but there is a shift in phase of ϵt_1, explaining the difference in the location of the sharper peak for above and below resonant frequencies.

Comparison of Figure 20 with Figure 22 (or Figure 21 with 23) shows the wave form is essentially the same at both chamber ends implying that the wave form is similar throughout the chamber. The amplitudes are greater at the nozzle end than at the injector end since at the nozzle there is no phase or delay in the reflection but there is a phase in reflection at the injector end. This phase in reflection is related to the time-lag effect. Note that if a finite length nozzle were considered there would be a phase in reflection at the nozzle entrance related to the wave travel time within the convergent portion of the nozzle.

Figure 20, 21, 22, and 23 show a very slight difference between the second and third order approximations implying convergence for small values of ϵ . For larger values of ϵ , the difference between the three approximations is significant. Also for larger values of ϵ , a double-valued solution occurs at the sharper peak indicating shock formation. This is strikingly different from the case of shock formation for a simple wave which originally had a sinusoidal wave form. In that case, shock formation is well-known to begin at the inflection point of the compressive portion rather than at the peaks. This seems to be a result of off-resonant oscillation.

The qualitative results for the higher mode of oscillation ($\ell = 2$) are identical. The only important difference involves, of course, the period of oscillation.

CHAPTER IV

DISCUSSION OF RESULTS

ANALYTICAL RESULTS

An important result of the analyses of Chapter II and III is the indication of the importance of the characteristic time of the combustion process. In Chapter II the combustion time is negligible compared to the wave travel time in the chamber such that the energy feedback from the combustion process to the wave phenomenon is instantaneous. On the other hand, in Chapter III, the combustion time and the wave travel time are of the same order of magnitude such that the energy feedback is not instantaneous but occurs over a period of time of the order of the wave travel time.

Instability of the steady-state operation occurs in both cases if the feedback of energy is sufficiently strong. In Chapter III, the further the ratio of period of oscillation (approximately wave travel time) to combustion time (time-lag) is from the optimum value (2), the greater must be the feedback (or, in other words, n must be greater) in order to produce instability. At the optimum value of this time ratio, energy is fedback in phase with the pressure oscillation. Away from this optimum time ratio the phase between pressure and energy addition or feedback becomes nonzero such that the absolute value of the energy addition must become larger. In Chapter II the energy addition is instantaneous so that the phase is always zero and thereby optimum for instability.

Whenever the combustion time and the wave travel time are of the same order the frequency of the oscillation is different from the natural frequency of the chamber by an amount of order u_o. Furthermore, unstable operation is possible over a certain chamber-length range such that upper and lower length stability limits exist. Whenever the combustion time is negligible compared to the wave travel time, the frequency of oscillation is the natural resonant frequency of the chamber and instability is possible at all chamber lengths provided that the energy feedback is sufficiently strong. Note, however, that the assumptions of concentrated combustion zone at chamber and short nozzle become poorer as chamber length decreases. In fact, as more of the combustion occurs nearer the pressure node (at the chamber center for the fundamental mode) the operation becomes more stable. The result is that in practical cases a lower length limit exists so that the only claim made herein for instantaneous combustion processes is that no upper length limit exists.

Stable periodic solutions of finite amplitude without shock waves could only be found if the phasing between energy addition and pressure were sufficiently far from zero. Whenever the phasing were too close to zero the periodic solutions without shock waves (Chapter III) were found to be unstable. It may be argued that the only stable periodic solutions in this case will contain shock waves. If the amplitude is increased above the value for the unstable periodic solution, this amplitude will continue to increase. If a stable periodic solution exists, this growth must be stopped. This growth can only be stopped by the appearance of one or more additional dissipative mechanisms since the nozzle is not sufficient in this case to prevent the growth. With growth of amplitude, there is a distortion of the wave form. With sufficient growth and distortion, shocks are formed providing the additional dissipative mechanism.

Note that the above statements concerning the relationship between phase and solutions with or without shock waves result from an analysis wherein the phasing effects appear due to the Crocco time-lag effect. It is not clear that these statements can be generalized to include other types of combustion phenomena where phasing is present. The statements are in agreement with the results of the analysis of Chapter II in which the limiting case of zero phase is treated. In that case all finite-amplitude oscillations contained shocks*. It is felt, therefore, that this result is valid even for combustion models not based on the Crocco time-lag postulate.

Even if the phasing between energy addition and pressure is far from zero, the amplitude must not be too large if no shocks are to form. If the amplitude is so large that severe distortion of the sinusoidal wave form occurs, shock waves may form. So, solutions without shocks are found only in a small

* Shocks were assumed to exist in that analysis; however, their amplitudes were left to be determined. If solutions with no shocks were possible, zero shock amplitude solutions could have been found.

region in the τ_o , n plane. This region is adjacent to the $\tau_o^{(o)}$ vs. $n^{(o)}$ curve but far from the minimum value of $n^{(o)}$ (point of zero phase between energy addition and pressure). The width of this region is of order $\bar{\epsilon}^2$ where $\bar{\epsilon}$ is the upper bound on the value of the amplitude parameter such that shocks do not form.

A very important result is the relationship between the forcing function of the instability and the wave form in the chamber. This is readily seen from the results of Chapter II where the nature of the forcing function is described (to sufficient accuracy for our purposes) by the values of the parameters ω and δ. Here, we see that the amplitude is directly proportional to $(\omega-1)$ while the coefficients of x and t appearing in the exponentials are each directly proportional to δ. Therefore, a knowledge of the forcing function (i.e., knowledge of ω and δ) leads to a prediction of the amplitude of the oscillation and the shape of the wave form.

The interesting possibility here is that the amplitude and wave shape may be determined experimentally and the theory may be used to calculate an ω and a δ. In this way, something can be learned about the nature of the forcing function. This point will be exemplified later when the Princeton gas rocket experiments are discussed.

This same type of relationship has been found in Chapter III where the wave form of the stable periodic solutions is found to depend upon the combustion parameters* τ_0, n, M, and L. It is seen that $s, \epsilon, A, \theta, C_p$, B, and Δ depend upon these parameters. The last five quantities depend upon the zero order values, $\tau_0^{(0)}$ and $n^{(0)}$ while the first two quantities ϵ and s depend not only on $\tau_0^{(0)}$ and $n^{(0)}$ but also upon the displacement from the neutral line or, in other words, upon τ_0' and n'.

This relationship between the combustion parameters and the wave form in Chapter III is extremely complex so that one probably can not determine the parameters by experimental observation of the wave form in this case as was suggested for the case of Chapter II. Even if the relationship were not so complex, there are still two other serious difficulties. The first is that stable periodic solutions are found only outside the range of n-values of practical interest. Stable solutions where wave form calculations are meaningful occur for $n > 2.0$ while according to Ref. 2, $n < 2.0$ for practical cases. Other injector-propellant combinations not yet tested may provide higher values of n, but this is not probable. The second serious difficulty is that n, M, and L appear in the relationships in a specific manner. In particular, the description of the combustion process is obtained by means of the Crocco time-lag postulate. As already mentioned, this postulate has been verified experimentally for the linearized

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* Other parameters such as γ, u_o, and the mode of oscillation affect the wave form, also.

treatment (Ref. 1 and 2). The extension of this postulate to the nonlinear case, however, has not been verified. No efforts have been made in this direction so that we are not certain that n, τ_0, M, and L appear in the nonlinear terms in a physically accurate manner.

The possibility of "triggering action" has been indicated by the instability of the periodic solutions in certain ranges of the τ_0 vs. n plane. Whenever the periodic solution is unstable, an oscillation greater in amplitude than that of the periodic solution (by any amount, no matter how small) grows in amplitude until another dissipative mechanism presents itself. The implication is that if a disturbance is induced with an amplitude greater than that of the periodic solution, this disturbance grows to some regime state such that unstable operation of the engine results. If the amplitude of the induced disturbance is below this critical value, the amplitude decays to zero and steady-state operation of the engine is restored. This critical value of the amplitude varies as the square root of the displacement from the neutral line in a τ_0, n plane. (This is an asymptotic relationship and may only be applied in a small region near the neutral line.)

The results for all cases where numerical calculations were performed indicate that "triggering" action is possible in the approximate range $n < 2.0$ which is the range of physical interest. It is noted with caution, however, that these results are based upon the nonlinear extension of the time-lag theory and this extension has not been verified experimentally. The author feels, however, that the results justify the claim that "triggering" action is a very likely possibility in the longitudinal mode whenever a phase exists between energy addition oscillation and pressure oscillation. In the zero-phase case studied in Chapter II, no "triggering" action was possible. Finite amplitude waves occurred only for values of ω which were in a range that produced instability of the steady-state operation to small perturbations (see, also, Appendix C). In Chapter III, the possibility of "triggering" action at the point of zero phase ($n^{(o)} = \frac{\gamma+1}{4\gamma}$) is seemily indicated since D is positive there. This would contradict the results of Chapter II except that accurate interpretation of the results of Chapter III leads to agreement rather than contradiction.

Consider a constant normal displacement at each point along the neutral line. This means $\epsilon^2 D$ is constant and finite. Since D becomes positive infinite as the zero-phase point ($n^{(o)} = \frac{\gamma+1}{4\gamma}$)is approached, ϵ must vanish. This would seem to mean that at this point any disturbance above the minimum amplitude $\epsilon = 0$ would grow in amplitude and, therefore, "triggering" action is extremely easy. In fact, it is easier at this zero-phase point than anywhere else where $\epsilon > 0$. On the contrary, $\epsilon = 0$ does not mean zero amplitude for the complete wave form but only zero amplitude for the lowest harmonic in that wave form. In order to have zero amplitude of the wave form, the amplitudes of all of the harmonics in that wave form must be zero. If any of the amplitudes $\epsilon^2 A, \epsilon^3 B,$ etc. are different from zero, the amplitude of the wave form is different from zero.

Note that, for a constant normal displacement, ϵ is inversely proportional to the square root of D and the amplitude of the higher harmonics are directly proportional to A/D, $B/D^{3/2}$, etc. In addition, note that both numerator and denominator become infinite at the zero-phase point as shown by Figure 14, 17, and 19. The numerical results indicate that these amplitude factors do not go to zero as the zero-phase point is approached but instead increase in magnitude. In fact, $B/D^{3/2}$ tends towards infinity as shown by Figure 24.* Hopefully, an asymptotic analysis (in which the limit as $n^{(0)}$ goes to $\frac{\gamma+1}{4\gamma}$ is considered) would show that the amplitude of one or more of the higher harmonics becomes infinite at this point. This would imply that the amplitude of the disturbance necessary to grow in amplitude rather than decay would be infinite. "Triggering", therefore, would be impossible at this zero-phase point and agreement with the result of the analysis of Chapter II is self-evident.

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* In Figure 24, the absolute value is plotted. The reason that $|B/D^{3/2}|$ becomes infinite at approximately $n^{(0)} = 2$ is that the displacement is changing from outward to inward as $n^{(0)}$ increases such that D is passing through zero. B remains finite, however. Since the displacement from the neutral line is equal to $D\epsilon^2$ plus terms of order ϵ^3, this means that the third order terms are important in determining the relationship between displacement and amplitude.

This asymptotic analysis would present a tedium comparable to that of the analysis of Chapter III and, for this reason, it has not been performed. Instead, it is assumed that the numerical results for points near the minimum value of $n^{(0)}$ $(=.46$ for $\gamma=1.2)$ give reliable information concerning the asymptotic behavior. It is claimed, therefore, that at least the amplitude of the third harmonic $\epsilon^3 B$ tends towards infinity even though the amplitude of the first harmonic ϵ tends towards zero. Furthermore, on this basis, the minimum amplitude of the disturbance necessary to "trigger" unstable operation of the engine must tend towards infinity and, in the limit, "triggering" is not possible.

These conclusions concerning the "triggering" action apply only to situations where there exists a "continuity" between the linear and nonlinear mechanisms of the energy addition to the oscillation. The essential physics of the response of the combustion process to various pressure and velocity disturbances, whether infinitesimal or finite, must be similar. An example of a "discontinuity" in this mechanism would be the droplet shattering phenomenon whereby velocity disturbances larger than a certain magnitude would cause inertial forces to become greater than surface tension forces resulting in the break-up or shattering of droplets into smaller droplets. The vaporization rates and burning rates increase causing an increase of the energy fedback to the oscillations. If the droplets are extremely small after shattering, the feedback of energy to the oscillation is essentially instantaneous and there is zero-phase. Under circumstances such as the one outlined here, "triggering" action may be possible at the zero-phase point.

It is noteworthy that if there is a "continuity" between the linear and nonlinear mechanisms of energy addition to the oscillations, there is a "continuity" in the results of a linear and a non-linear analysis. As $\epsilon \longrightarrow 0$, the nonlinear results of Chapter III and the linear results of Ref. 1 agree. Furthermore, the same stability limits are predicted by the nonlinear analysis of Chapter II and the linear analysis of Appendix C.

The importance of the boundary conditions in the determination of the stability criterion cannot be overemphasized. Growth of the amplitude of oscillation occurs whenever more energy is added by the combustion process than is withdrawn by the nozzle. Decay of the amplitude occurs whenever less energy is added by the combustion process than is subtracted at the nozzle. The nozzle is extremely important in determining the stability criterion. In fact, it is just as important as the combustion process and therefore its effects must be given an accurate description in any accurate analysis of the combustion instability phenomenon.

Crocco has developed a description of the nozzle effects under oscillating conditions, but the linearized treatment applies only at very small amplitudes (see Ref. 1). At larger amplitudes, the proper nozzle boundary condition is known only in the limiting case of zero-length convergent portion of nozzle. This limiting case has been assumed here. (Note that zero length really means negligible length compared to chamber length.)

Although a limiting case has been considered, the physics of that situation has accurately been described. Various other workers in the field have refused to consider limiting cases and, finding themselves without any knowledge of the proper boundary conditions, have been forced to make unjustifiable and (to this author's contention) physically unreasonable assumptions concerning the chamber flow field. In Reference 19, a sinusoidal pressure variation with time at a point slightly upstream from the nozzle throat was assumed while in Reference 20, a shock wave was assumed to have a constant pressure ratio with time during its travel down the chamber. In Reference 21, which contained an analysis of the transverse mode of instability, the longitudinal gradients were assumed to be zero. The need for simplification in treating these problems is clearly understood; however, it is hoped that any simplifying assumptions are proven reasonable before they are applied. Satisfactory proof should at least consist of a comparison of the results of the analysis based upon the simplifying assumptions with the results of an analysis based only upon assumptions which are widely-accepted as reasonable. This comparison could only be made for special limiting cases (otherwise there is no use for the questionable analysis if a clearly-reasonable analysis could be performed for a general case).

In Chapter II, where an oscillation at the natural resonant frequency of the chamber was studied, a second order analysis was necessary to obtain a first order result. This is common procedure in the analysis of nonlinear ordinary differential equations which have periodic forcing functions at the natural frequency (see Ref. 18). This procedure was also used in Ref. 11 and 12 where shock-wave oscillations in a one-dimensional chamber closed at both ends were treated. In Chapter III, where off-resonant oscillations were considered, it was not necessary to go to a higher order analysis to get the results of a given order. This is common procedure in nonlinear ordinary differential equations which have forcing functions at other than the natural frequency. Note that the governing equations for x and t are inhomogeneous wave equations which are analoguous to ordinary differential equations with forcing functions.

The amplitude of the resonant solution of Chapter II was directly proportional to the displacement from the neutral stability line ($\omega - 1$). The amplitude of the off-resonant solution of Chapter III was directly proportional to the square root of the displacement from the neutral stability line (represented by n' and τ_0').

"Sawtooth" wave forms were obtained in Ref. 11 and 12 and "nearly sawtooth" waves were found in Chapter II indicating that "sawtooth" waves are probably the natural types resulting from the wave phenomena. Ref. 17 discusses how sawtooth waves result asymptotically from other wave forms so that it is not surprising to obtain these results. The effect of the combustion process seems to be to provide small corrections on this sawtooth wave form as shown by Figure 7.

In Chapter III, stable wave forms without shock waves were obtained only at values of $\tau^{(0)}$ and $n^{(0)}$ quite far from the zero phase point (far from resonant frequency) and only for very small displacement from the neutral line (ϵ small). The wave forms were similar throughout the chamber. There were significant differences in the wave forms for above-resonant and below-resonant cases as explained in Chapter III and indicated in Figures 20, 21, 22, and 23. The values of $n^{(0)}$ for which stable wave forms are obtained seem, however, to be too large to be in the range of practical interest.

It is not possible to compare the results of these analyses with other analyses concerned with the nonlinear aspects of combustion instability since other theories involve the determination of flow conditions based on certain initial conditions (see Ref. 19, 20, and 21). Numerical integration is usually required so that results are not readily interpreted. If those analyses would yield periodic solutions it would occur only after an infinite time. Here, on the other hand, periodic solutions have been found by analytic means.

Comparisons can be made with works on related phenomena, however. The longitudinal oscillations with shocks studied by Chu (Ref. 11 and 12) are comparable to the oscillations studied in Chapter II. The similarity in the wave forms has already been mentioned. In a sense, Chu's problem is a limiting case of our analysis if the mean flow goes to zero and the injector end and nozzle end become solid walls. This is not expressible explicitly since as the mean flow goes to zero in our case, the energy addition also goes to zero and there can be no periodic oscillation. An important difference results due to the presence of the nozzle which removes energy from the oscillation. In Chu's case where there was no nozzle, the energy addition at one chamber end is of order ϵ^2 (where ϵ is the amplitude of the oscillation). In our case with a nozzle, both the energy addition due to combustion and the energy removal by the nozzle are of order ϵ, but their difference (net energy addition) is of order ϵ^2.

The results of Chapter III are similar to the results of Maslen and Moore (Ref. 9 and 10) even through their analyses dealt with transverse oscillations. Both here and in their works, no shock waves were considered with the second order effect being the addition of a constant* plus a second harmonic term to the wave form and the third order effect being the addition of a third harmonic term to the wave form. No phase appeared

* This means constant in both time and space for our case but for the case of Maslen and Moore, it means constant in time through variable with chamber radius.

in these higher order terms in their case as they did in our case. Another similarity is that their nonlinear frequency correction was of second order as is this term in our case.

Certain simplifications involving the application of the characteristic coordinate perturbation technique have been introduced herein. The first involves the use of the Riemann invariants rather than the gas velocity and the speed of sound even though the boundary conditions are given in terms of these latter properties. In addition to their invariance along the characteristics, these former variables possess the convenient property of continuity (up to and including second order) through shocks. A similar convenience is obtained by use of the variables $\left[\frac{x_1}{1-u_0}+t_1\right]$ and $\left[\frac{x_1}{1+u_0}-t_1\right]$ rather than x_1 and t_1. These former quantities are continuous through a shock wave. In similar fashion to the Riemann invariants, only the combination $\left[\frac{x_1}{1-u_0}+t_1\right]$ is continuous across forward-moving shock waves while the combination $\left[\frac{x_1}{1+u_0}-t_1\right]$ is continuous across reaward-moving shock waves.

Another simplification was introduced in Chapter III whereby a certain degree of arbitrariness was allowed in the numbering system for the characteristic coordinates. This allowed the determination of a transformation $x(\alpha,\beta)$ and $t(\alpha,\beta)$ that possessed a continuous functional form for all values of α and β. This type of transformation simplified the process of determining periodic solutions. Even if $u(\alpha,\beta)$ and

$a(\alpha,\beta)$ are periodic, it is not certain that $u(x,t)$ and $a(x,t)$ are periodic. In order to determine this $x(\alpha,\beta)$ and $t(\alpha,\beta)$ must be known over a wide range. It is more convenient for $x(\alpha,\beta)$ and $t(\alpha,\beta)$ to have the same functional form over the whole range of interest than to have different functional forms in different sections of the range. If there were no arbitrariness allowed in the numbering system, the latter situation would occur.

EXTENSION OF THE ANALYTICAL WORK

An interesting extension of the analytical work would be the determination of periodic solutions with shock waves whenever the energy feedback is described by means of the Crocco time-lag postulate. There is a serious difficulty which has prevented this extension.

According to the Crocco time-lag postulate, the perturbation in mass release of burned gases $\dot{m}'(t)$ is proportional to the pressure perturbation $p'(t)$ and is also proportional to the negative of the pressure perturbation at some well-defined prior instant $p'(t-\tau_0)$. This may be written as follows:

$$\dot{m}'(t) = n\left[p'(t) - p'(t-\tau_0)\right]$$

This is a linearized relation so that the error is of order ϵ^2.

τ_0 has been found to be approximately one-half the period* (or, at any rate, less than the period) for unstable situations. Now, the period of oscillation is the time between shock reflections at the combustion zone and, therefore, the time between discontinuities in $\dot{m}'$ and p' at the combustion zone. If a time τ_0 after a shock reflects is considered, it

*This statement could be generalized to include any odd multiple of one-half the period. The following argument would be essentially identical.

is seen that $p'(t - \tau_o)$ is discontinuous so that $\dot{m}'(t)$ and (or) $p'(t)$ must be discontinuous at this point. This implies there are other discontinuities in addition to those caused by the primary shock wave. All these discontinuous jumps would be of the same order of magnitude which is in serious disagreement with experimental results.

The employment of the time-lag concept which proved successful for the linear case results in failure for the non-linear case with shock waves. Apparently, this concept, as presently formulated, gives a good description of the combustion process in one instance but not so in another. Probably the basic physical concept is applicable in both situations; i.e., the characteristic time of the combustion process is an important parameter in the instability phenomenon. The formulation of this concept, however, would probably change as the flow and combustion process is modified; e.g., with the introduction of shock waves. It is concluded that until a better understanding of the combustion process is obtained, an investigation of the instability problem with time-lag effects and shock waves would not be fruitful.

It would be useful to relax the long chamber assumption and consider a distributed (longitudinally) combustion zone and a nozzle length which is comparable with the chamber length. Each of these relaxations would separately prevent the solution of the compatibility relations to be simply the Riemann invariants. Other terms involving the integrals of mass source and area change terms would appear making the solution much more difficult. The consideration of the degeneracy at the nozzle throat (where one family of characteristics is vertical in a time vs. space plot such that no signal is propagated back into the chamber from this throat position) would be quite difficult.

In conjunction with the relaxation of the above mentioned assumptions, it would be extremely useful to develop an approximate analytical technique in order to overcome the difficulty of the analysis instead of employing the technique used in this work. (The present technique is also approximate, of course, but the error is well-defined according to powers in ϵ). The merit of this other technique could be determined by applying it to the limiting cases of concentrated combustion at one chamber end and short nozzle. Those results could be compared to the results of this work. If the difference is small, the approximate technique which assumedly would be much simpler would be a very useful way to obtain results with reasonable accuracy.

A more accurate analysis of the combustion zone dynamics of Chapter II and Appendix B would require an accurate description of the rate function. For example, if chemical kinetics provided the rate-controlling factor in the energy release, one should use the rate $\rho^{\hat{n}} e^{-E/RT^*}$ rather than the rate e^{-E/RT^*} as used in the numerical example of Chapter II. The main difficulty here involves the determination of the gas density behavior under oscillating conditions. If the isentropic relation were used, the rocket would be unstable for values of $\hat{n} \geq 1$ regardless of the exponential effect. Stable operation over a certain range would be possible only for very small values of $\hat{n}$. However, energy release (and diffusion effects if diffusion time is not too long) makes the isentropic condition invalid within the combustion zone. The proper relationship could only be obtained by solution of the unsteady heat equation (which has not yet been performed for this situation).

COMPARISON WITH EXPERIMENTS

The best experiment for comparison with this theory of Chapter II seems to be the Princeton gas rocket. Details of that research will be presented in a report (Ref. 22) to be published shortly. The gas rocket is of variable length and small diameter and burns premixed (hydrogen-air) gases with E/RT* estimated at about 10, near the stability limits. Only the longitudinal mode of instability occurs and it is observed in the form of shock waves followed by small exponential decay in pressure. Figures 25 shows the pressure wave form observed with the Princeton gas rocket. The pressure jump across the shock is of the order of those calculated in the simplified example. At low lengths, the engine is stable, but at longer lengths, where the concentrated combustion at injector and short nozzle assumptions are reasonable, instability occurs (at the preferred mixture ratios) with no upper length limit. The preferred mixture ratios for instability occur away from and on both sides of stoichiometric such that the same temperature is measured at both stability limits. The various data collected in that program seems to be leading to the conclusion that chemical kinetics provides the driving mechanism for the instability.

If it is assumed that the gas rocket combustion process behaves in similar fashion to the model investigated in chapter two , and, more specifically, that the energy release rate r follows an Arrhenius type law approximated sufficiently well for our purposes by the numerical example, many of the observed phenomena can be explained by this theory. Using the estimated values of E/RT* for the gas rocket, the theory predicts instability consisting of shock waves followed by small exponential decay and also predicts the criticality of the temperature at the stability limits since the amplitude is linearly dependent upon E/RT*. The theory (which neglects friction dampening) predicts no upper length limit. The theory predicts instability at all lengths (for favorable temperature); however, the assumptions of the theory are violated at lower lengths. At short lengths, the combustion zone is nearer the pressure node of an acoustic oscillation which tends to be stabilizing and explains the lower length-limit of instability observed in the rocket.

In a series of tests performed by Crocco and Harrje (Ref. 6) using like-on-like injection in a liquid propellant rocket motor of variable length, results similar to those of the gas rocket were obtained if there were end-impingement of the spray fans of like propellants. That is, the instability regions occurred at low and high mixture ratios with no clear upper-length limit. (If there were no end-impingement, an external pulse was sometimes necessary to move the engine into unstable operation. The driving mechanism of the instability here might be suspected to involve droplet shattering.) The instability occurred at all mixture ratios of interest with no upper-length limit when the engine was pulsed. The instability occurred in the longitudinal mode and the wave forms were shock waves followed by exponential decays. Due to large mass flow per unit area, the amplitudes were quite high and it is questionable that a theory such as this one which involves a series expansion in an amplitude parameter could be applicable in a quantitative manner. However, qualitative comparisons might be possible on the basis of more definitive experiments.

Longitudinal instability consisting of shock waves followed by rapid decay of pressure has been observed in radially burning solid propellant motors by Brownlee (Ref. 7) and Dickinson (Ref. 8.) In these motors burning occurred over the length of the chamber. It is interesting, therefore, that even though the condition of concentrated combustion is violated, this type of instability is still observed.

One can see how for all types of rocket engines, a relationship should exist between the wave form in the longitudinal mode of instability and the energy release rate as a function of position in the chamber. Presently, it is possible to show this relationship analytically only for concentrated combustion at the chamber end and with zero phase between pressure and energy addition. In principle, though, this concep can be extended to other situations. However, for the sole purpose of seeking information concerning the combustion process when phase is negligible, an engine may be constructed which approximately satisfies the assumptions of the theory of Chapter II and the wave form of the instability may be observed by use of pressure transducers. This promises to be a powerful technique for the study of combustion processes. Of course, in applying this information to other configurations, it would be assumed that the combustion process remains unchanged. This approach might be applicable to end-burning solid rockets as well as gas rockets.

The validity of the results of Chapter II are presently being determined experimentally by means of the Princeton gas rocket research. As already mentioned, preliminary results are reported in Ref. 22. In addition, an experimental program is presently being undertaken to check the validity of the results of Chapter III.

The approach that is being used involved the experimental determination of the effect of "pulsing" the motor with disturbances of various magnitudes since in the region of practical values of n, the theory predicts the possibility of "triggering" action. The possibility of "triggering" action for the transverse case has been well established as shown by

Figure 2. Preliminary results for the longitudinal case are presented in Ref. 6, but, as already mentioned "triggering" action **may** involve the strictly nonlinear mechanism of droplet shattering. This analysis, on the other hand, assumes a continuity between the linear and nonlinear mechanisms.

CONCLUSIONS:

1) Unstable operation of the motor is possible for two situations. In one case, the characteristic time of combustion is negligible compared to the period of oscillation and, in the second, the two times are of the same order of magnitude.

2) It is strongly implied that the only stable periodic solutions of practical interest contain shock waves (in the longitudinal mode).

3) There is an important relationship between the forcing function of the oscillation and the wave form in the combustion chamber.

4) Observation of this wave form by experimental means could lead to certain information concerning the combustion process.

5) "Triggering" action of a longitudinal oscillation by finite disturbances seems most probable whenever there exists a phase between energy addition and pressure.

6) If there is a "continuity" between the linear and nonlinear forcing functions of the oscillation, there is a "continuity" between the linear and nonlinear results. For example, the same stability limits are predicted.

7) The nozzle boundary condition is extremely important in determining the stability criterion.

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Pressure waveforms: longitudinal mode
Chamber length=25 9/16"
Chamber pressure=300 psi

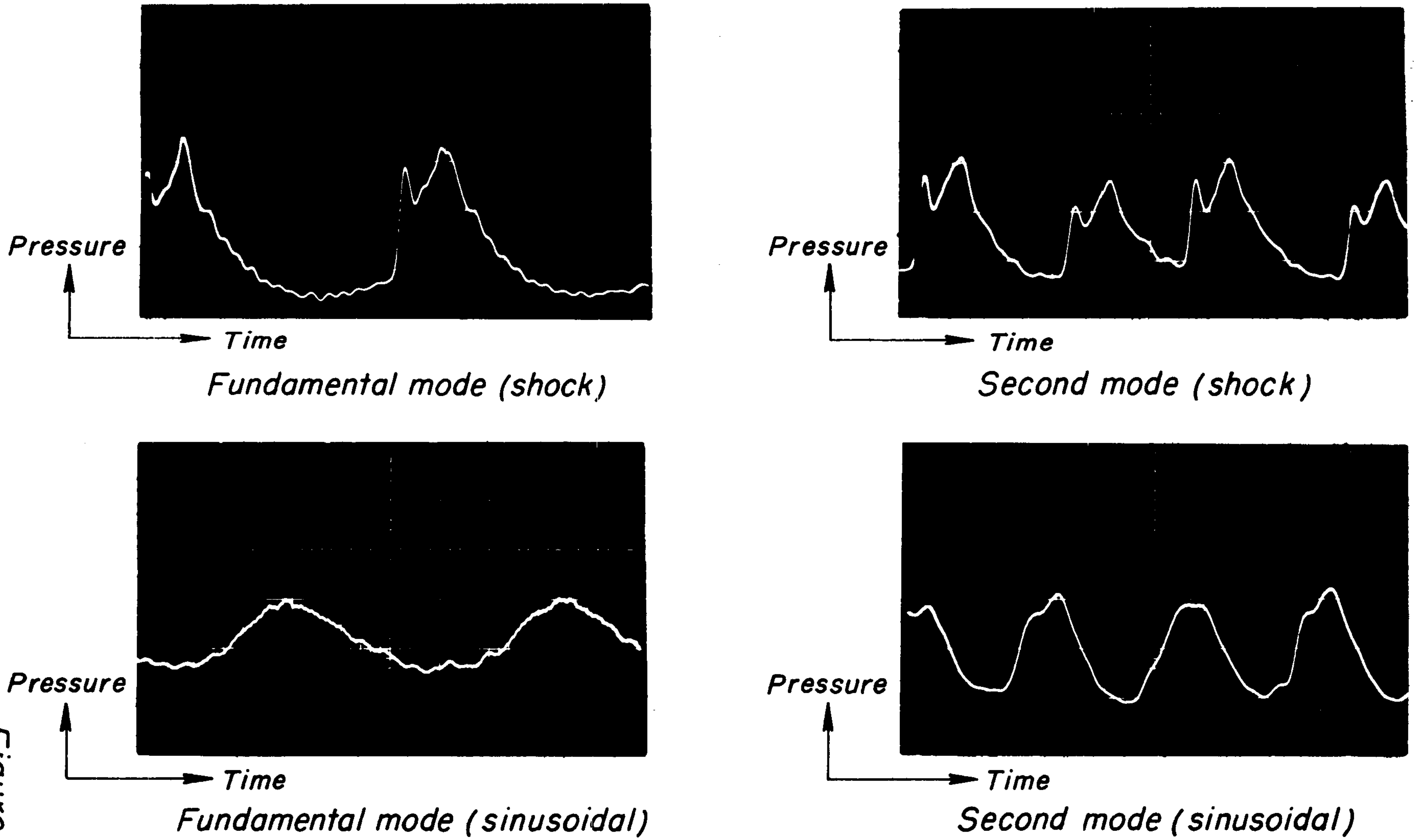


Figure 1

9-7 1.4 Tangential (1x12 spuds) 500 lbs thrust, 150 psi, pulsed instability limits tests

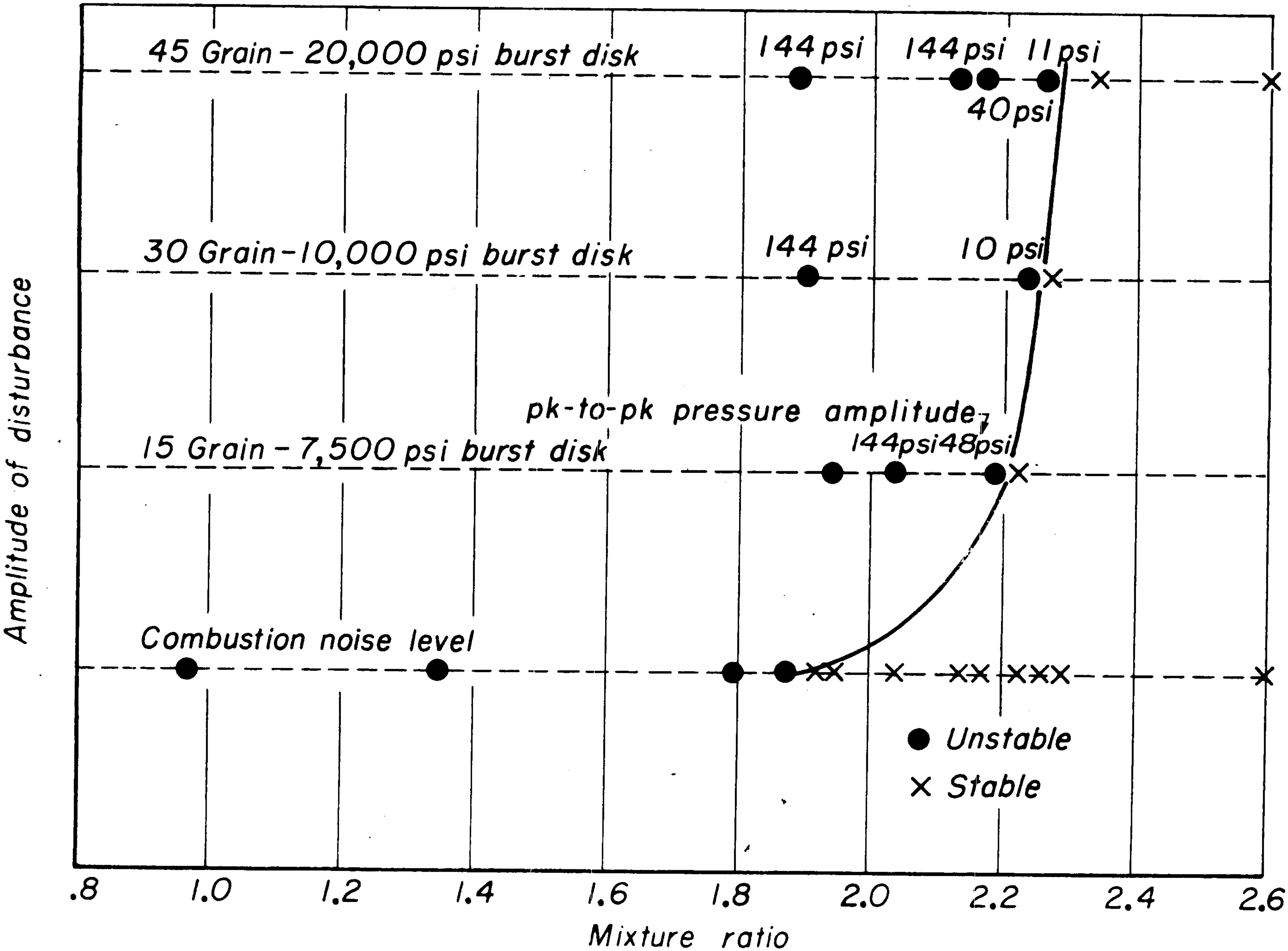


Figure 2

Coordinate numbering system

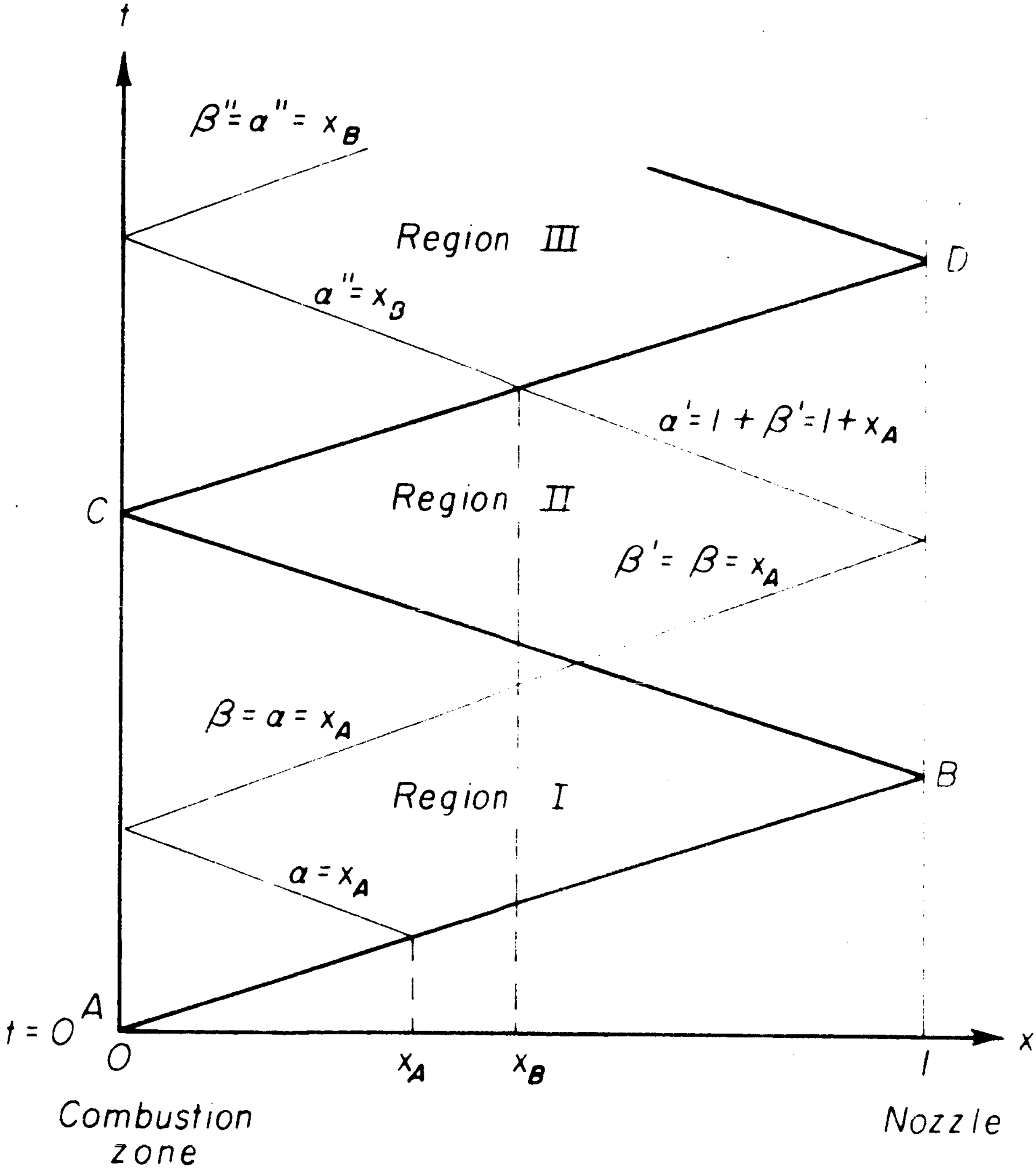


Figure 3

Continuation of solutions across shock waves

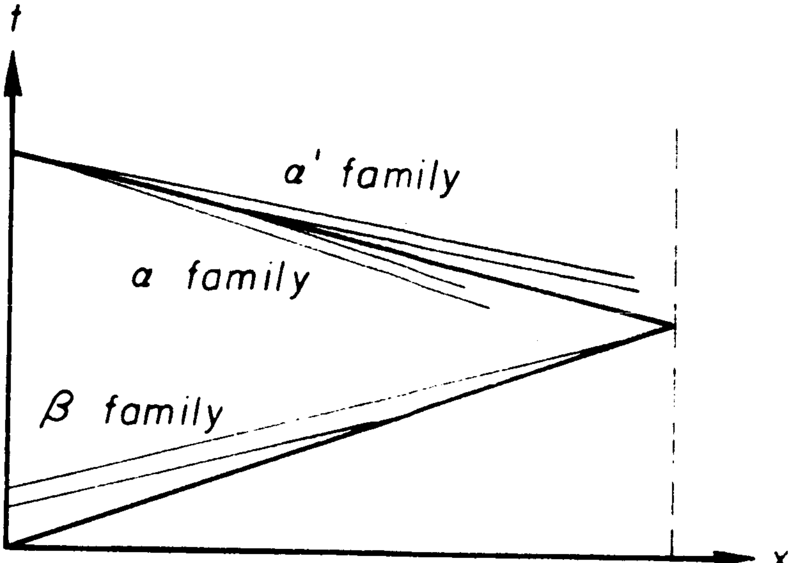


Figure 4a

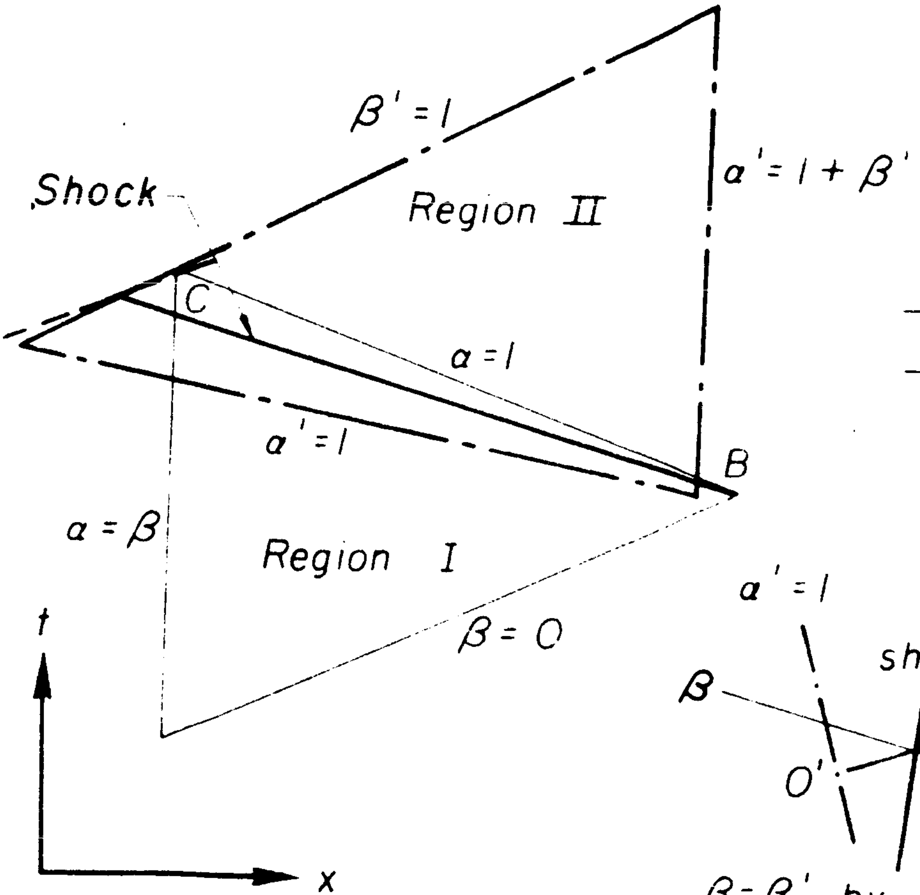
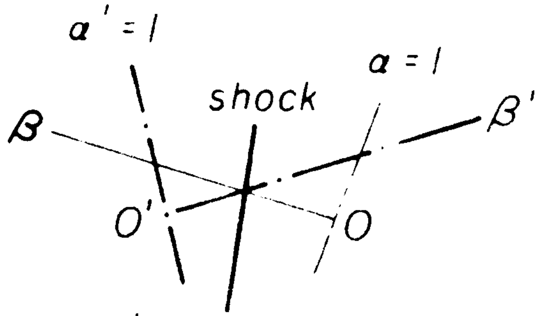


Figure 4b



β = β' by definition

Figure 4c

Figure 4

"Slippage" of characteristics

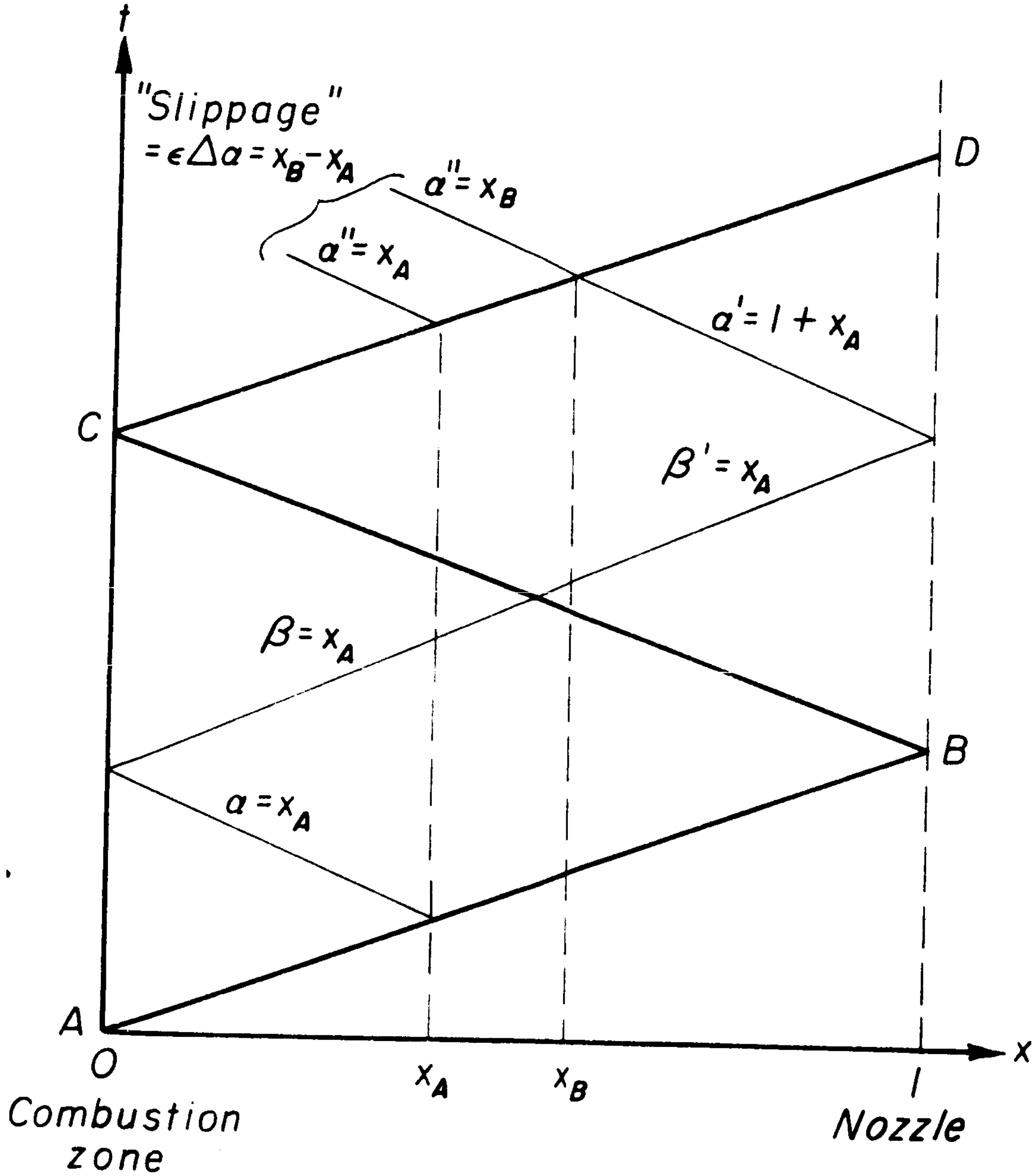


Figure 5

Wave shapes

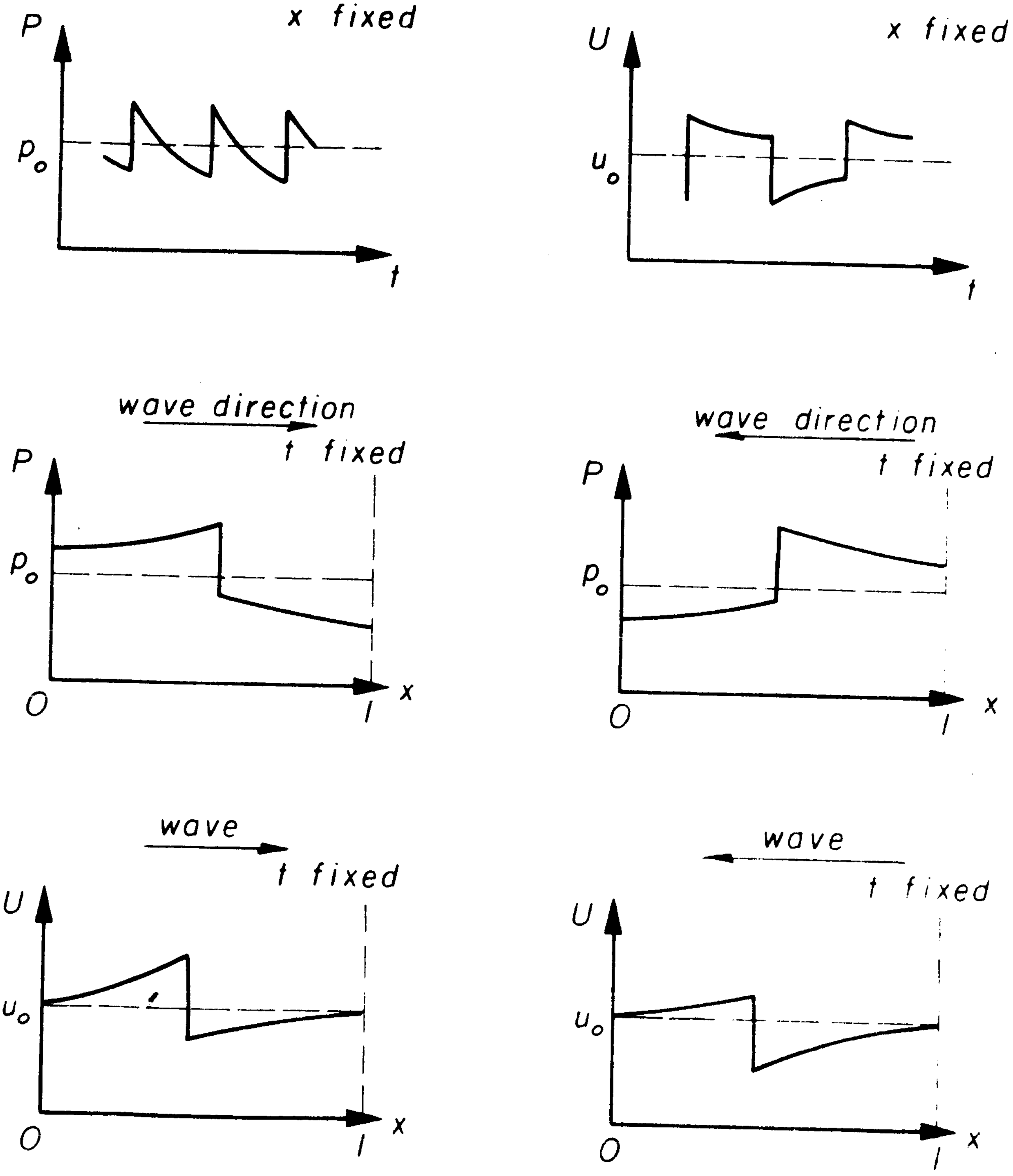
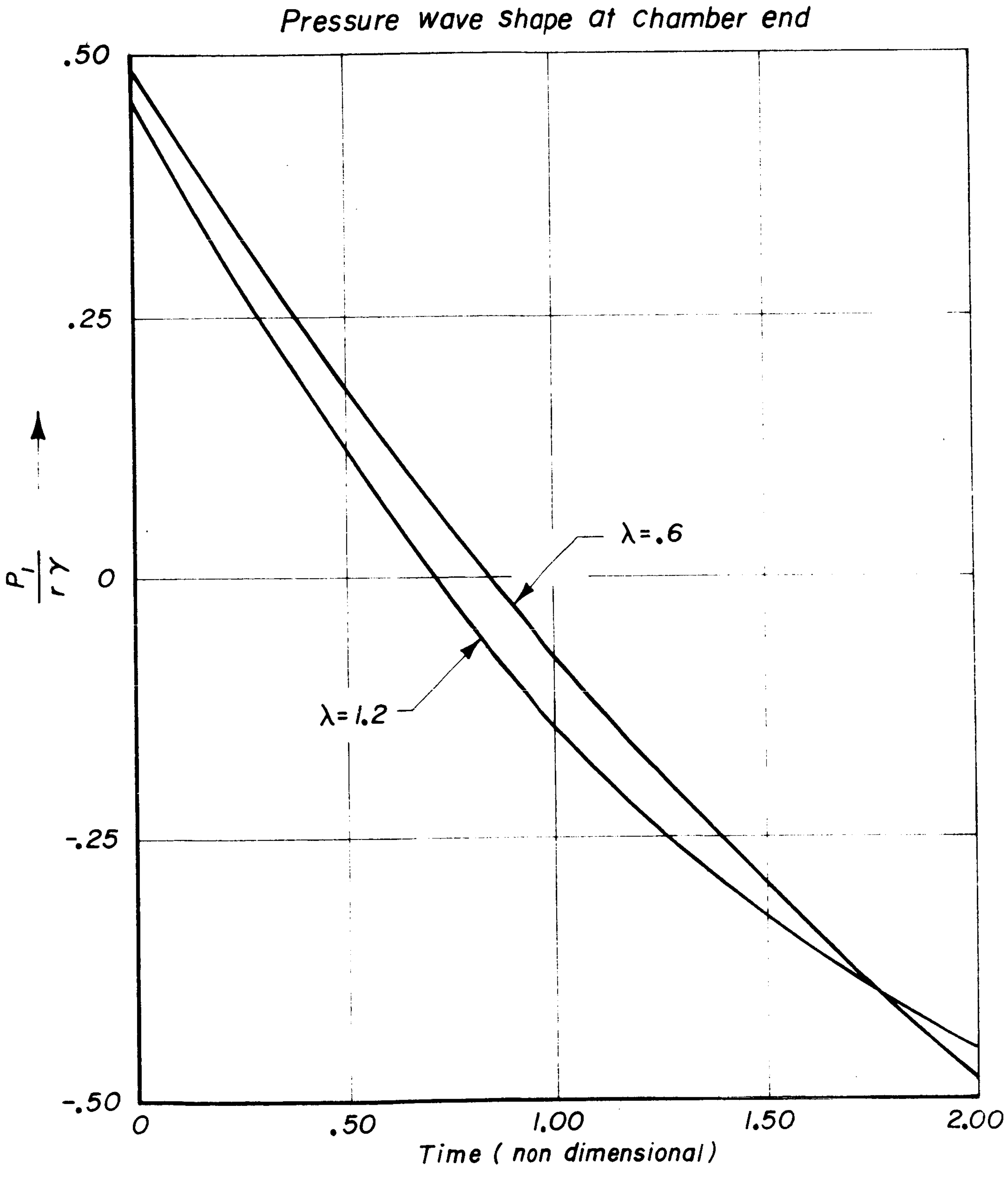


Figure 6



Pressure wave shape at chamber end
.50
.25
0
-.25
-.50
$\frac{P_1}{r\gamma}$
0
.50
1.00
1.50
2.00
Time (non dimensional)
λ=.6
λ=1.2

Figure 7

Coordinate numbering system

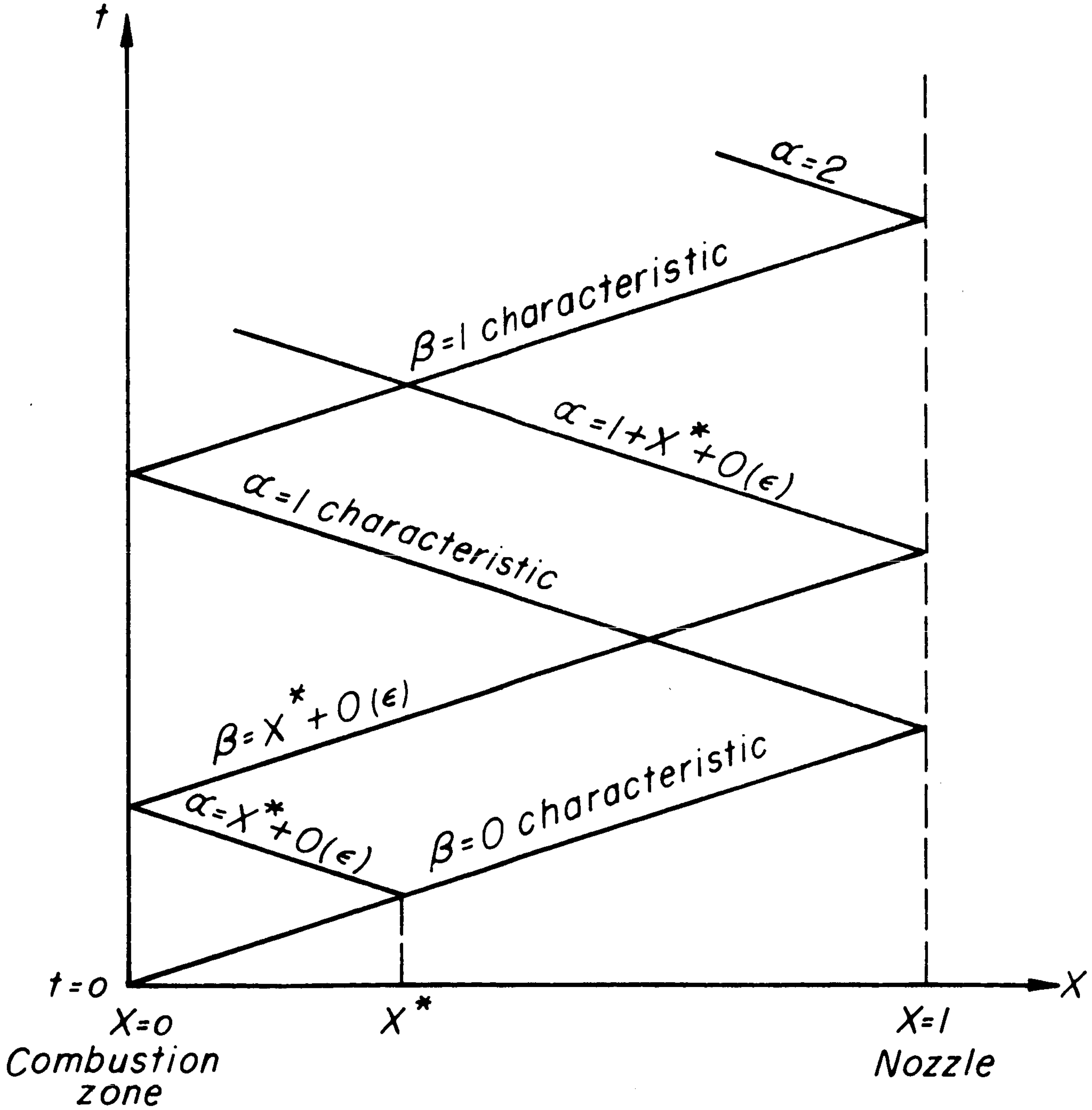
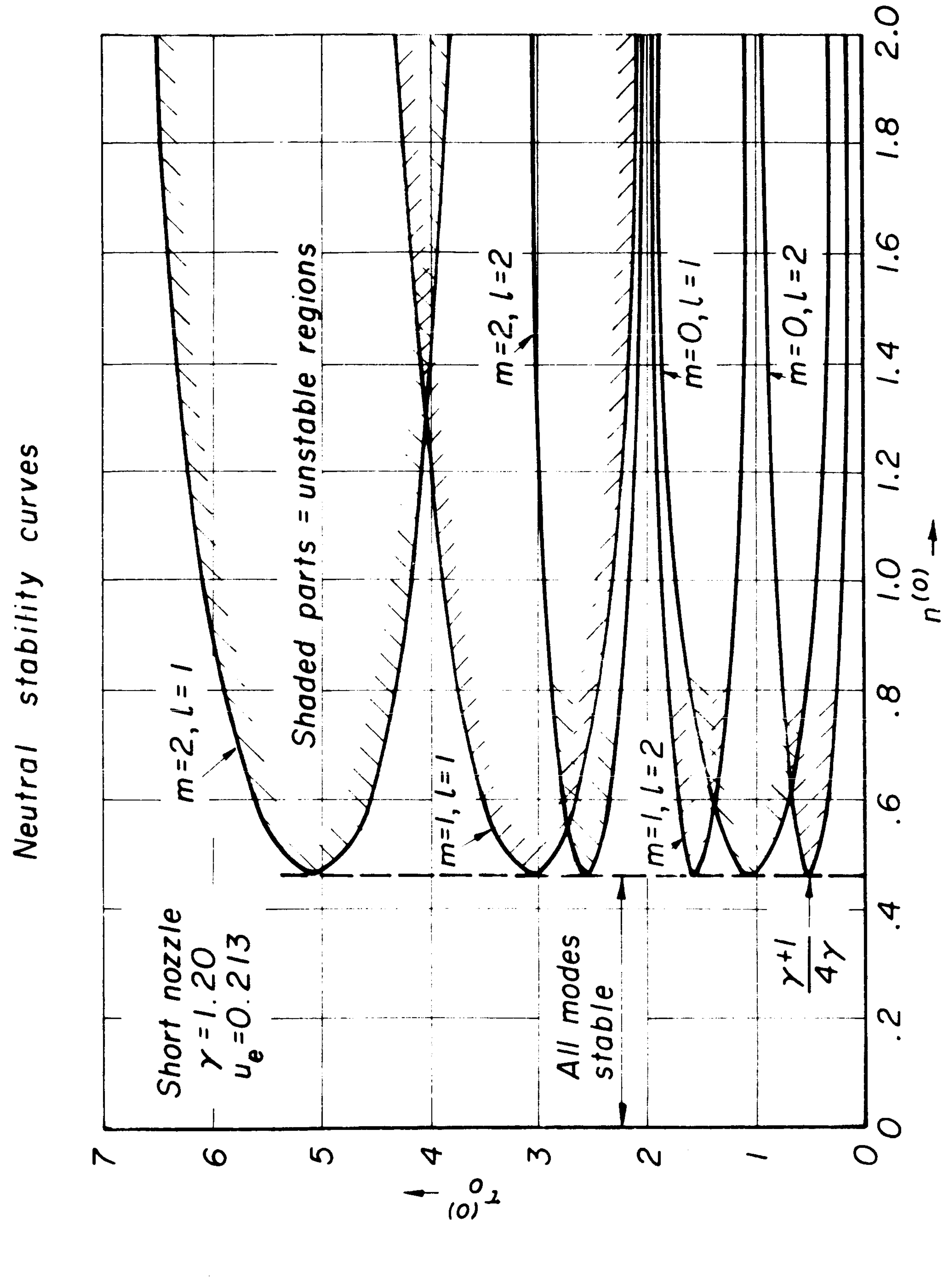


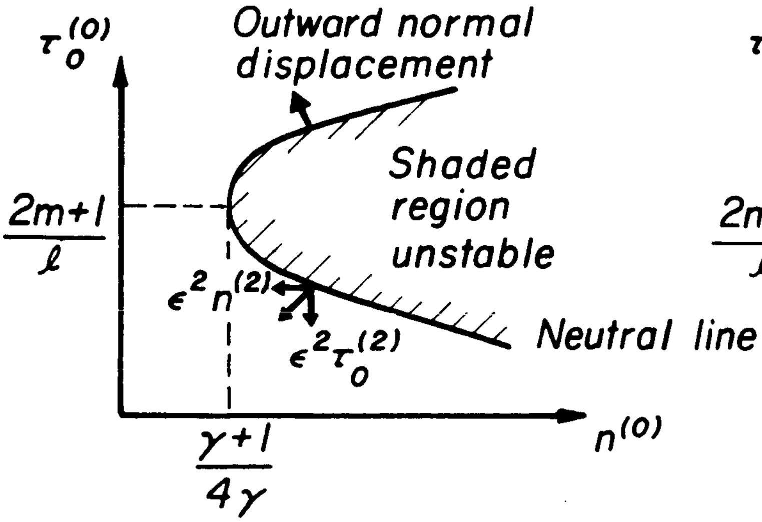
Figure 8



Neutral stability curves
Shaded parts = unstable regions
m=2, l=1
m=1, l=1
m=2, l=2
m=1, l=2
m=0, l=1
m=0, l=2
Short nozzle
γ=1.20
u_e=0.213
All modes stable
γ+1 / 4γ
n^(0) →
τ_0^(0) →
0 .2 .4 .6 .8 1.0 1.2 1.4 1.6 1.8 2.0
0 1 2 3 4 5 6 7

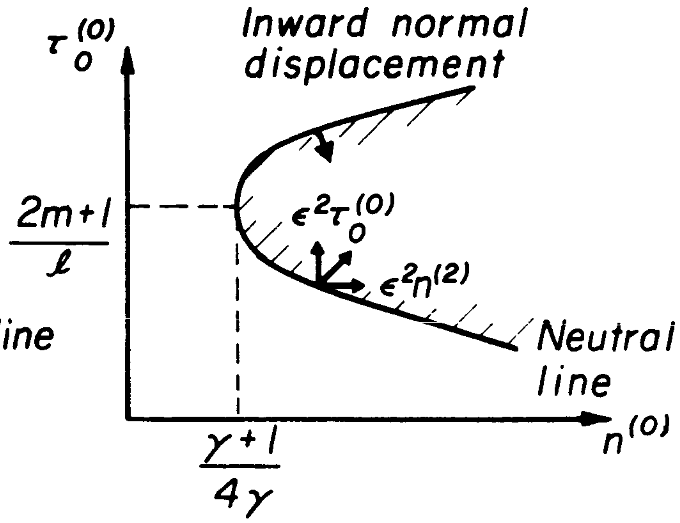
Figure 9

Effects of displacement from neutral line



$$\left[\ell_r \, y_2' - \omega_2' \, \ell_i \right] > 0$$

Figure 10a



$$\left[\ell_r \, y_2' - \omega_2' \, \ell_i \right] < 0$$

Figure 10b

Amplitude vs Displacement

$$\left[\ell_r \, y_2' - \omega_2' \, \ell_i \right] > 0$$

Unstable periodic solution

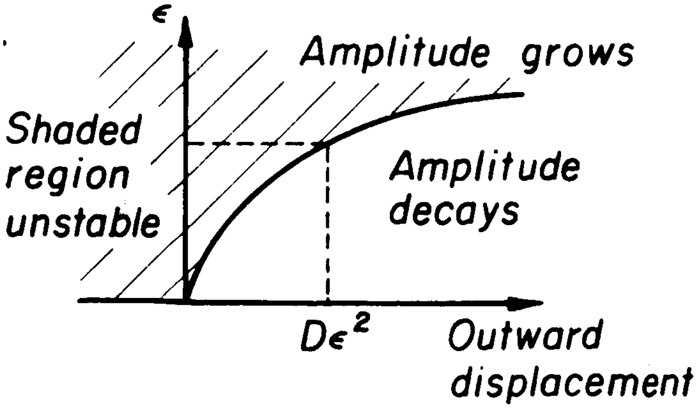


Figure 10c

Amplitude vs Displacement

$$\left[\ell_r \, y_2' - \omega_2' \, \ell_i \right] < 0$$

Stable periodic solution

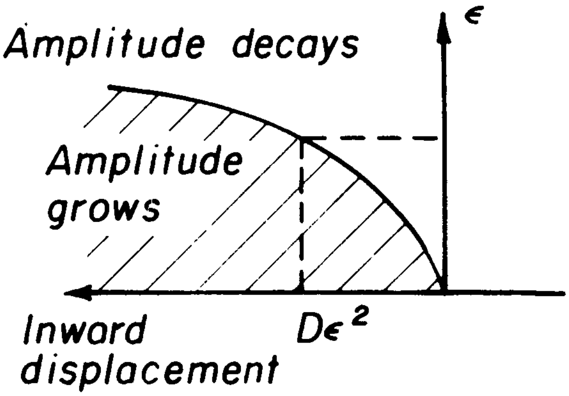
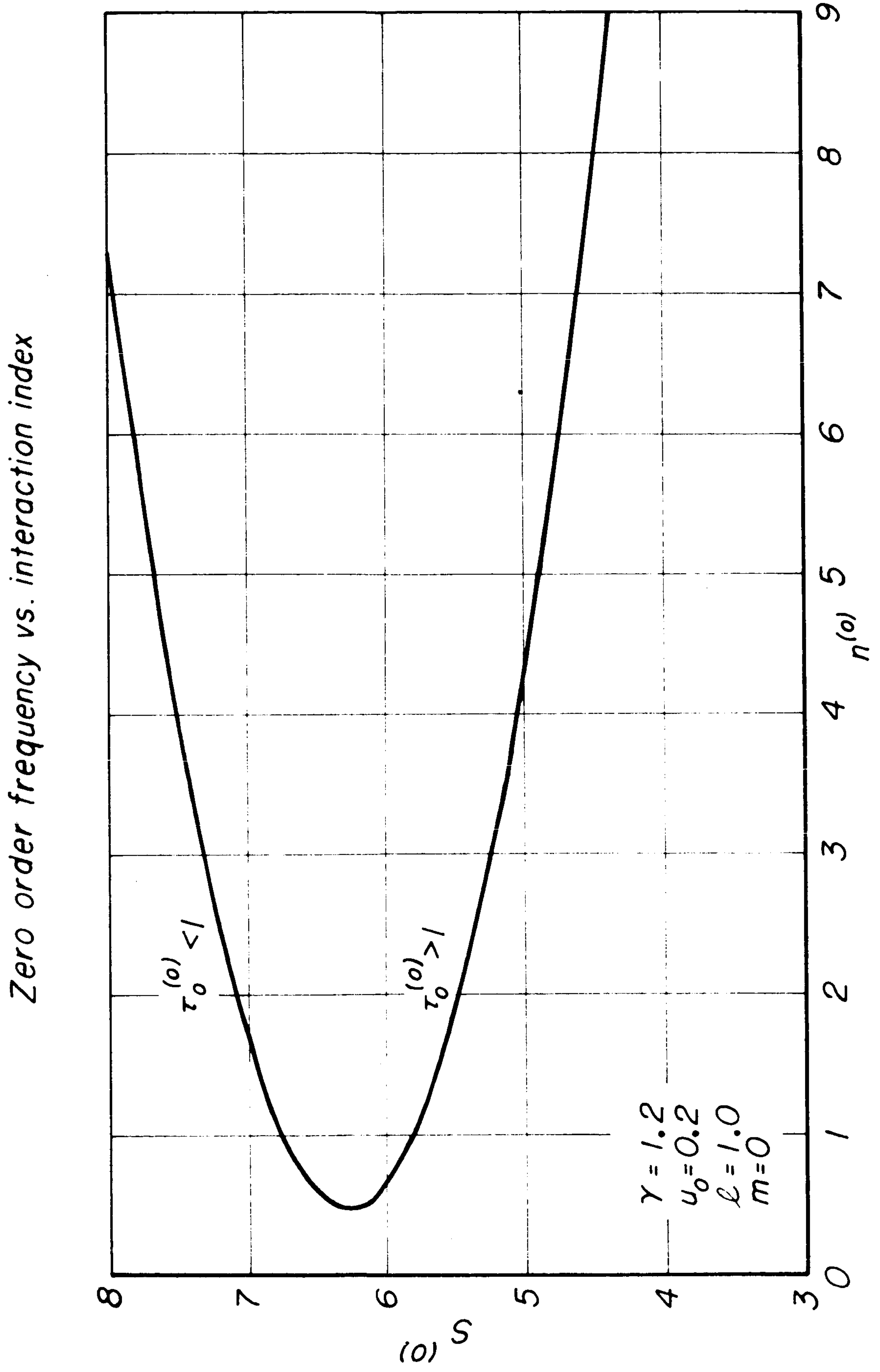


Figure 10d

Figure 10



Zero order frequency vs. interaction index
$\tau_o^{(o)} < 1$
$\tau_o^{(o)} > 1$
$\gamma = 1.2$
$u_o = 0.2$
$\ell = 1.0$
$m = 0$
$S^{(o)}$
$n^{(o)}$
0 1 2 3 4 5 6 7 8 9
3 4 5 6 7 8

Figure 11

Zero order characteristic coordinate lag vs interaction index

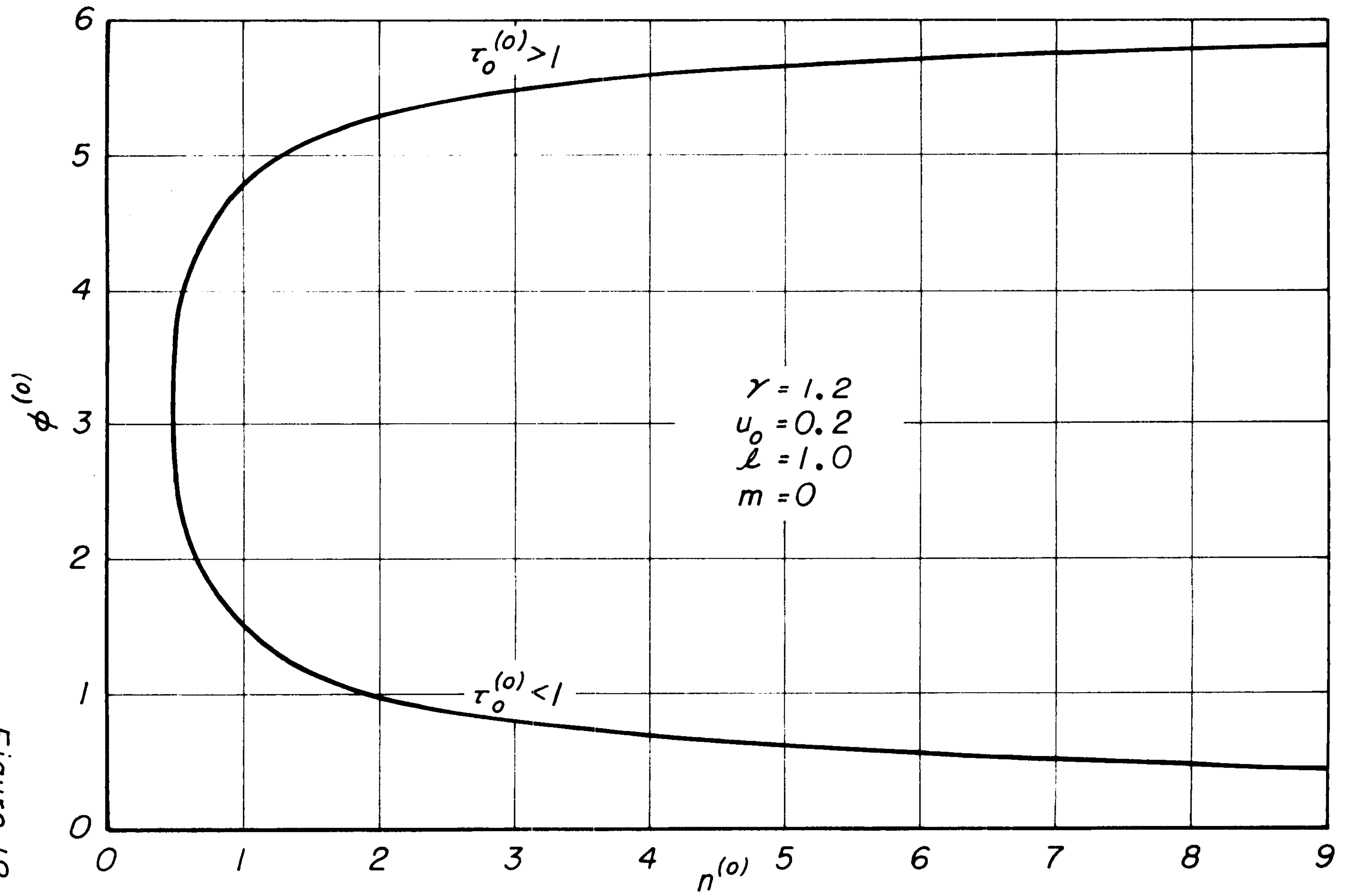
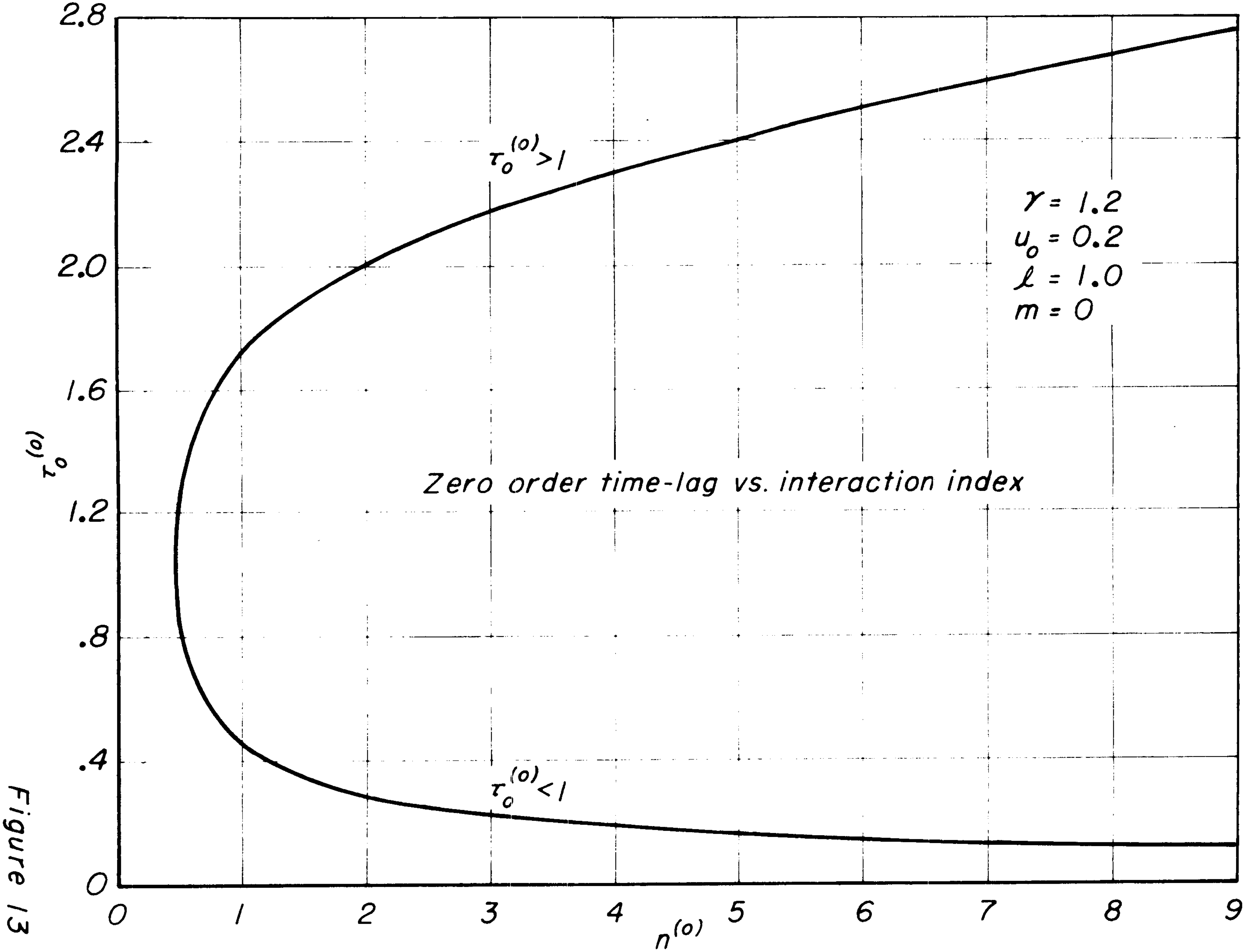


Figure 12



2.8
2.4
2.0
1.6
1.2
.8
.4
0
$\tau_o^{(0)}$
0
1
2
3
4
5
6
7
8
9
$n^{(0)}$
$\tau_o^{(0)} > 1$
$\tau_o^{(0)} < 1$
$\gamma = 1.2$
$u_o = 0.2$
$\ell = 1.0$
$m = 0$
Zero order time-lag vs. interaction index

Figure 13

Second harmonic amplitude factor vs. interaction index

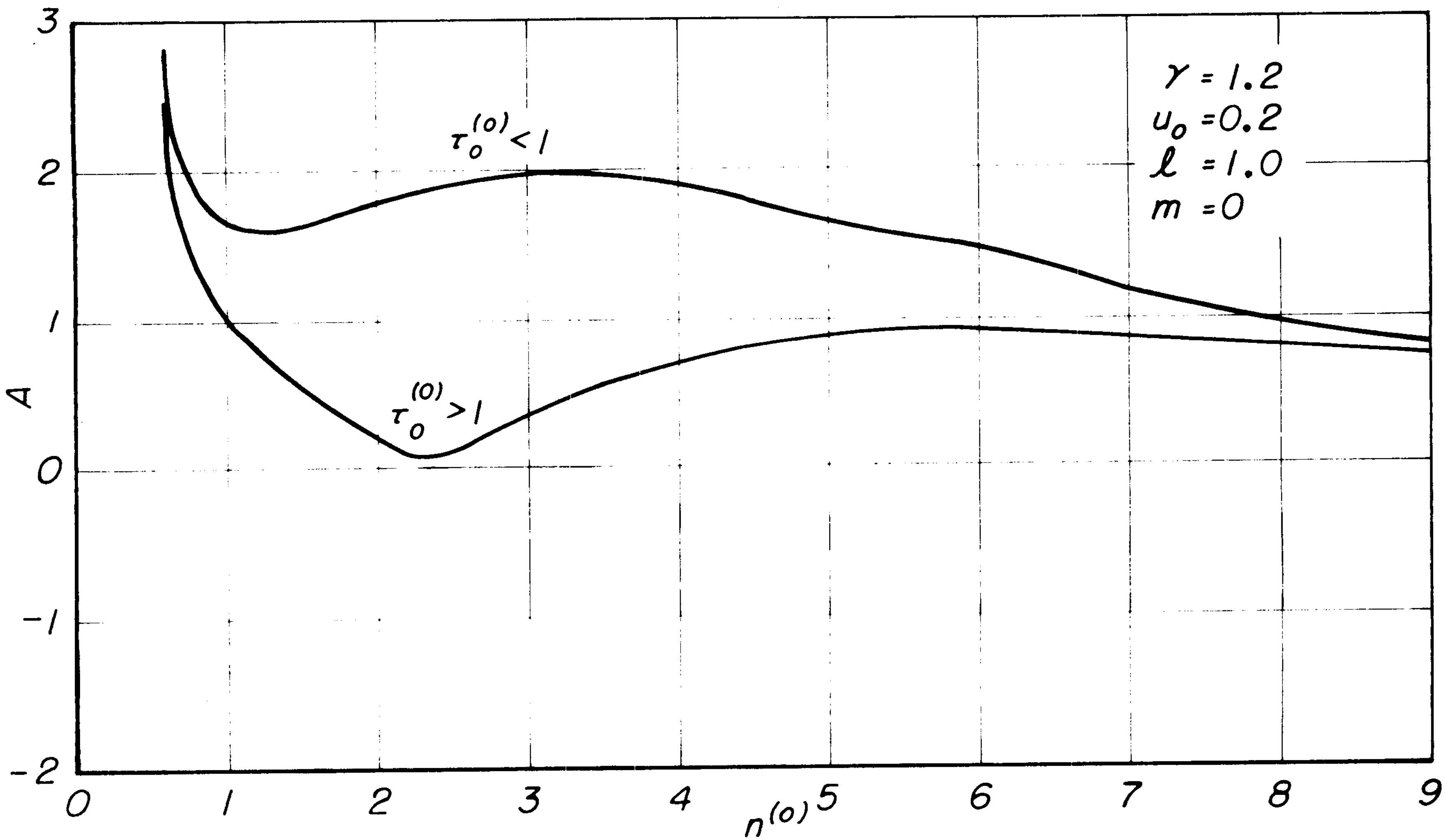


Figure 14

Second harmonic phase factor vs interaction index

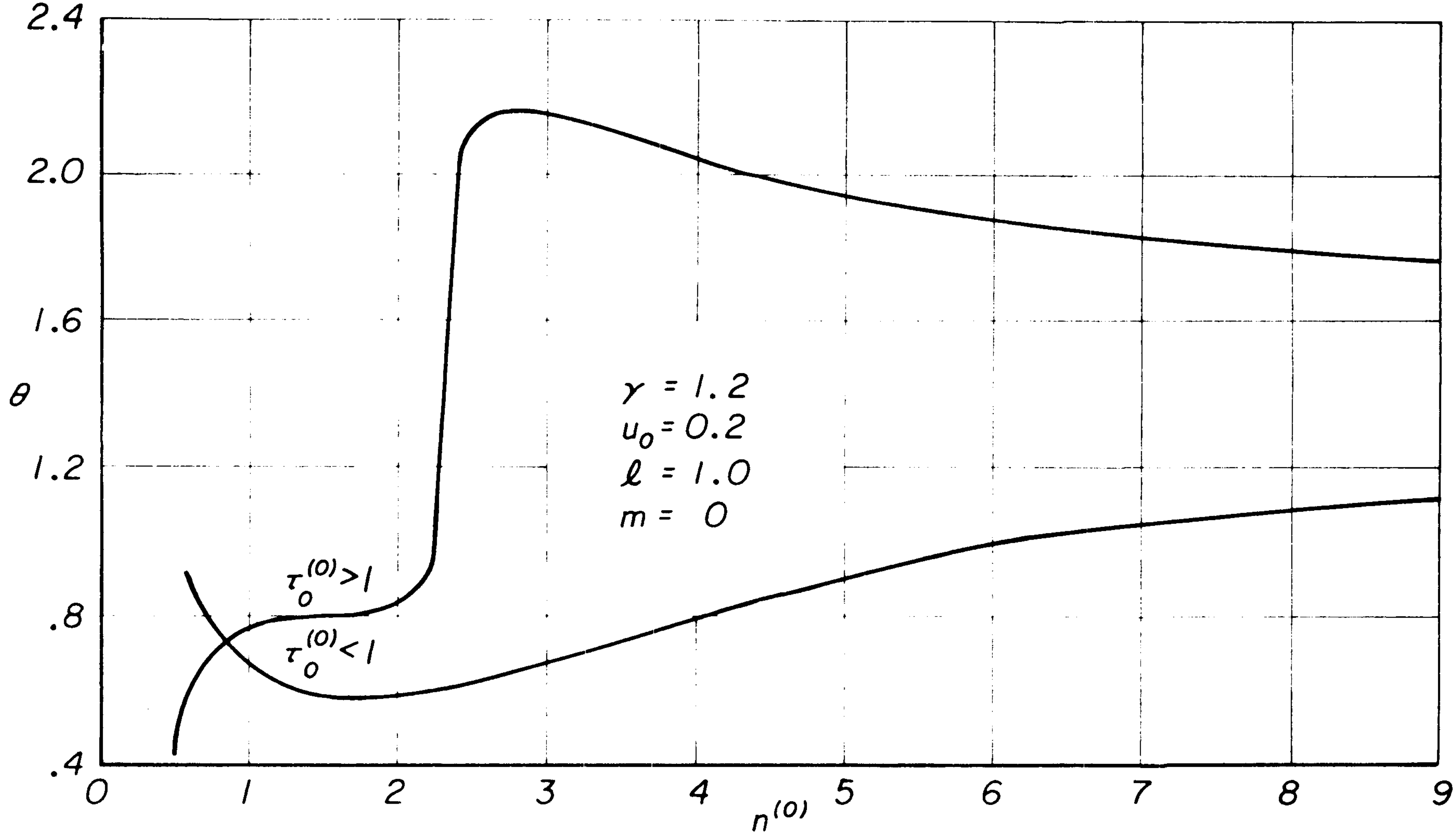


Figure 15

Second order mean flow corretion factor vs. interaction index

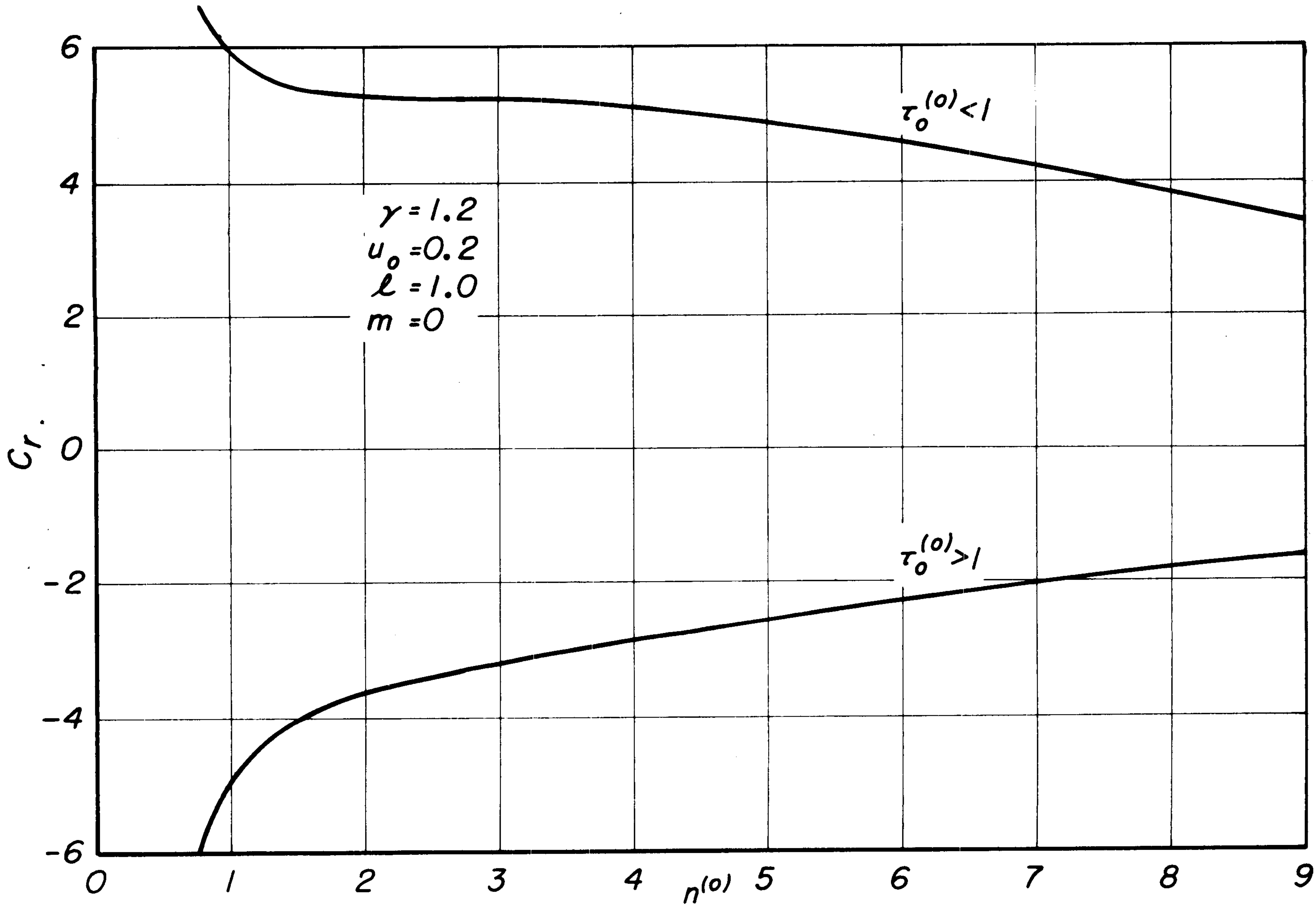


Figure 16

Third harmonic amplitude factor vs. interaction index

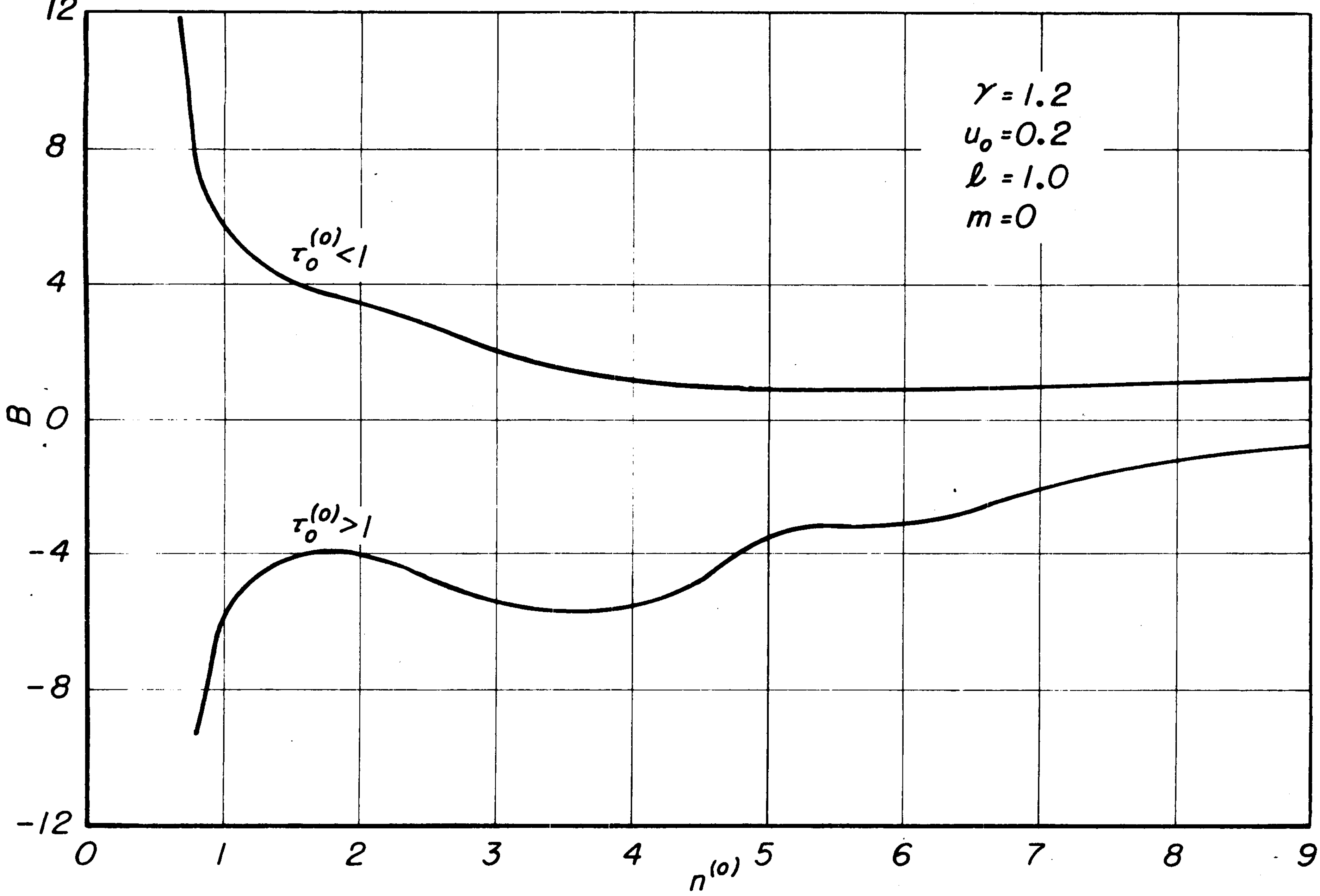


Figure 17

Third harmonic phase factor vs. interaction index

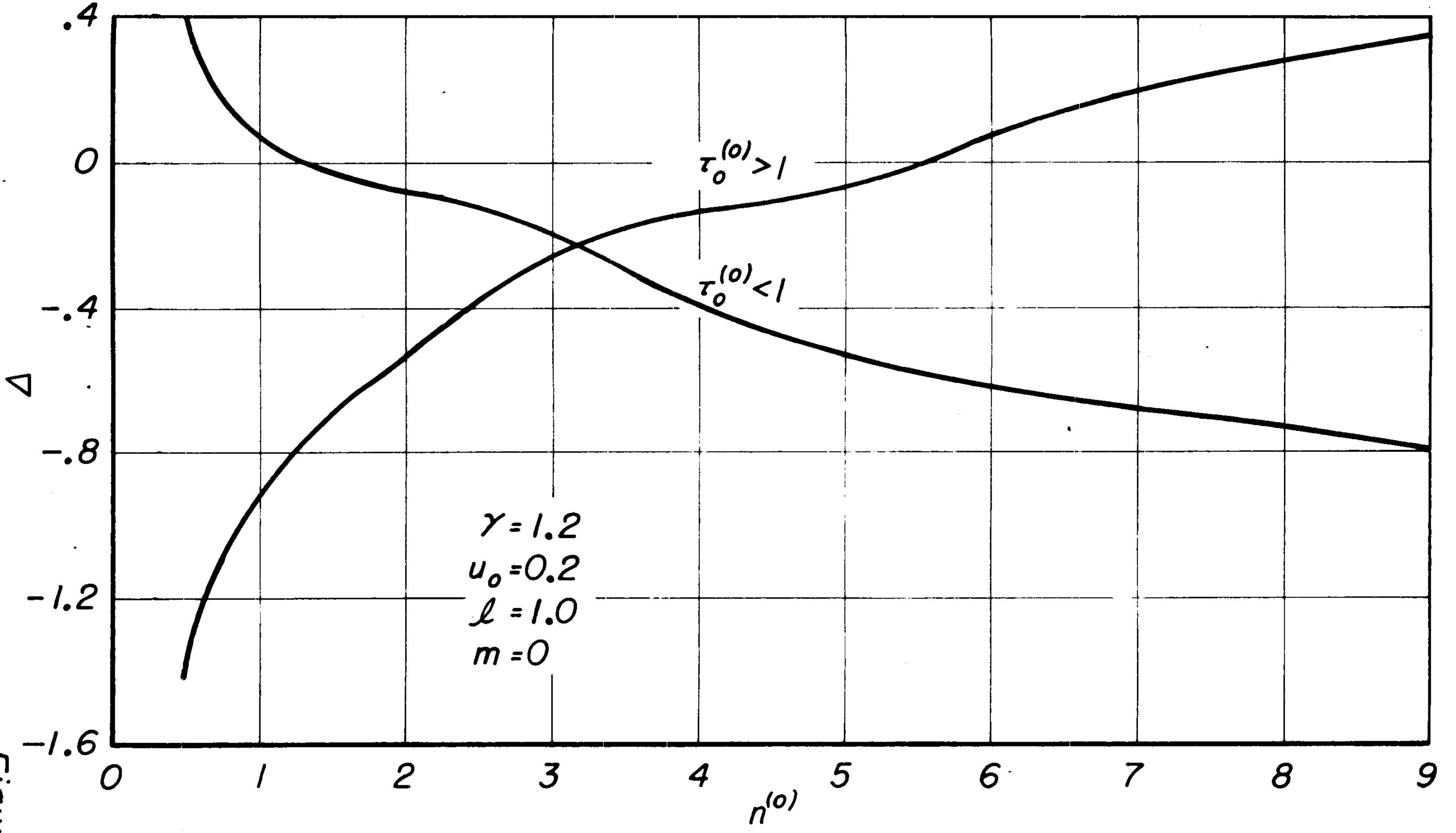


Figure 18

Neutral-line displacement factor vs. interaction index

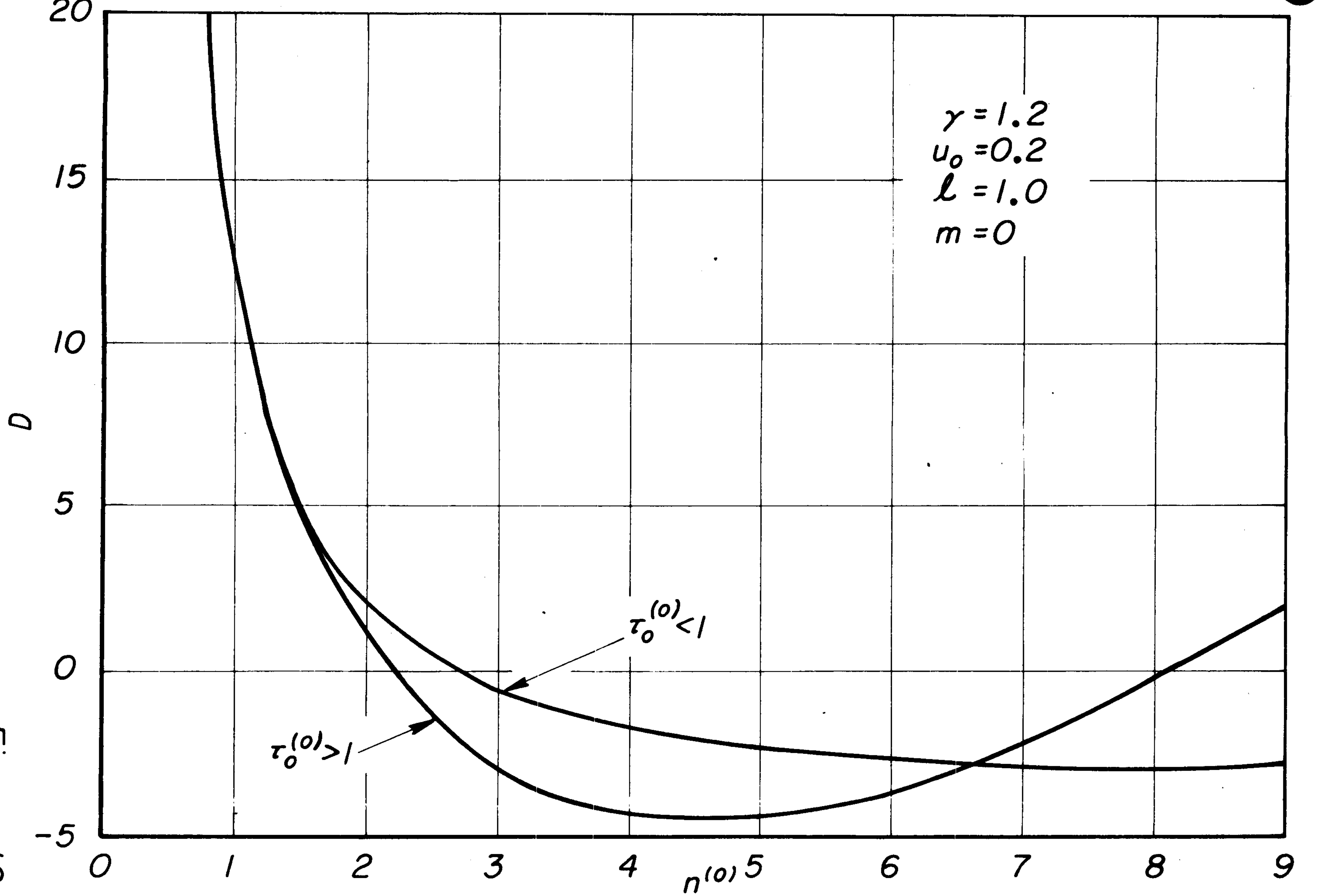


Figure 19

Pressure wave form at nozzle with above-resonant oscillation

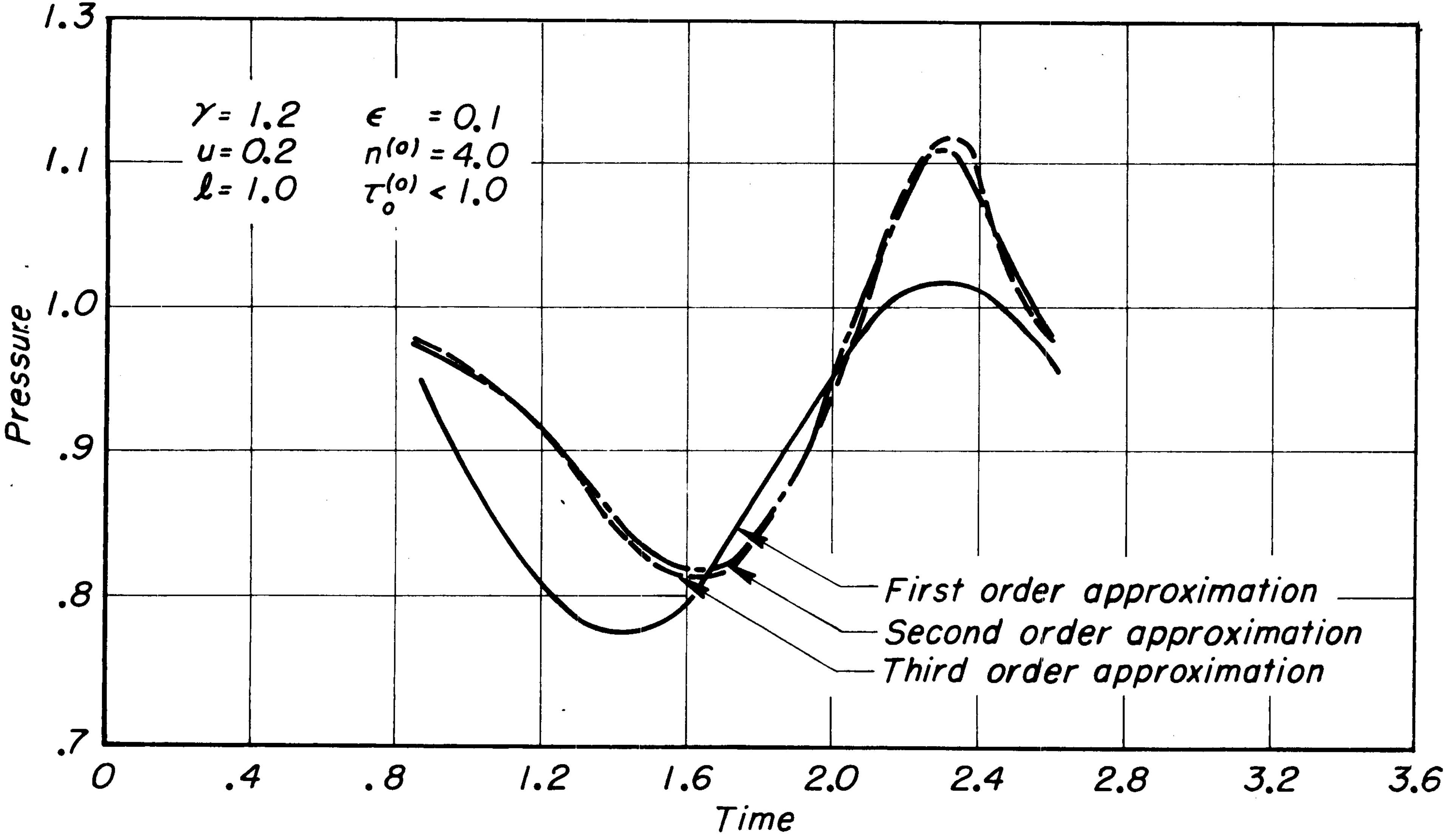


Figure 20

Pressure wave form at nozzle with below-resonant oscillation

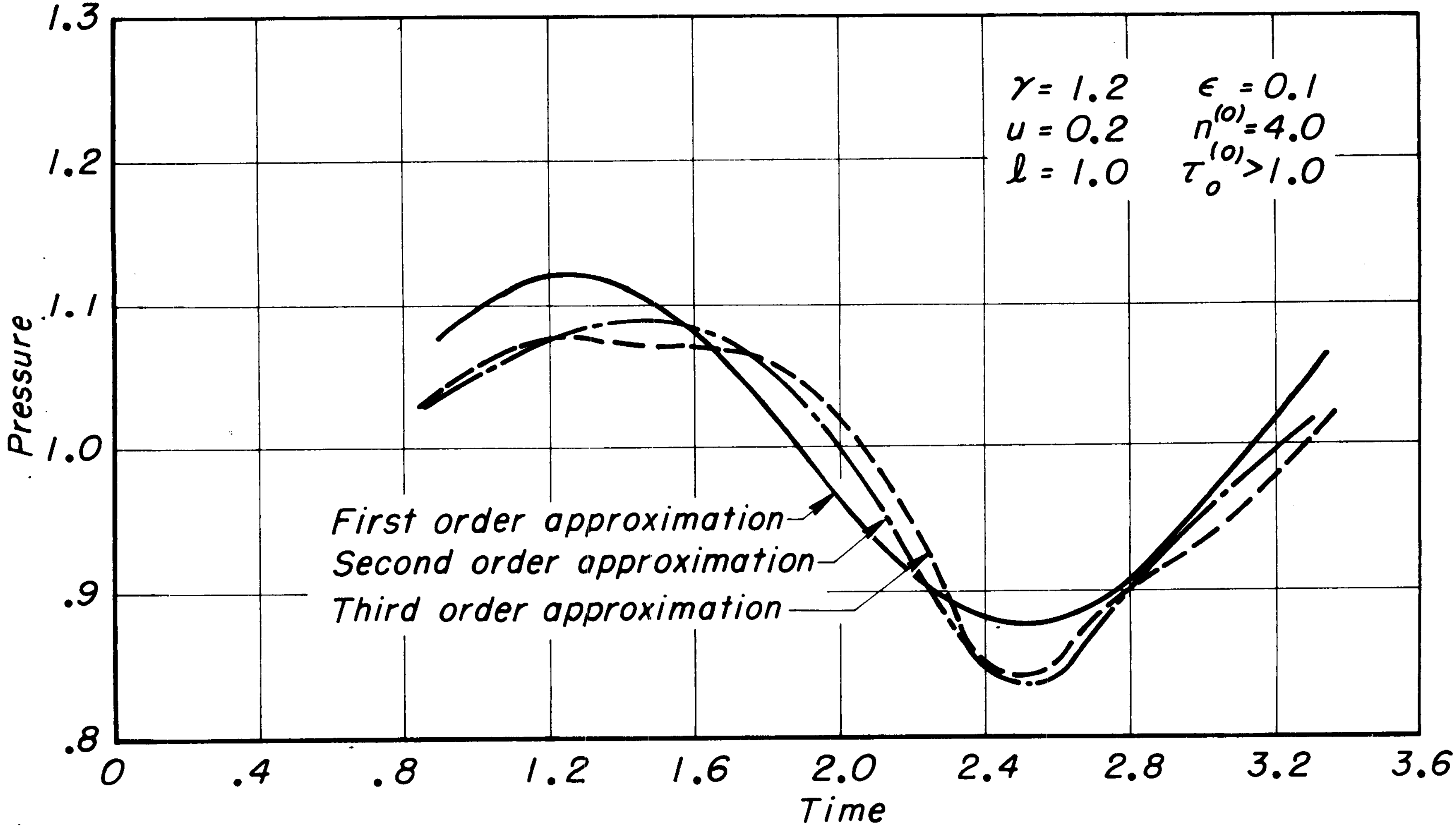


Figure 21

Pressure wave form at injector with above-resonant oscillation

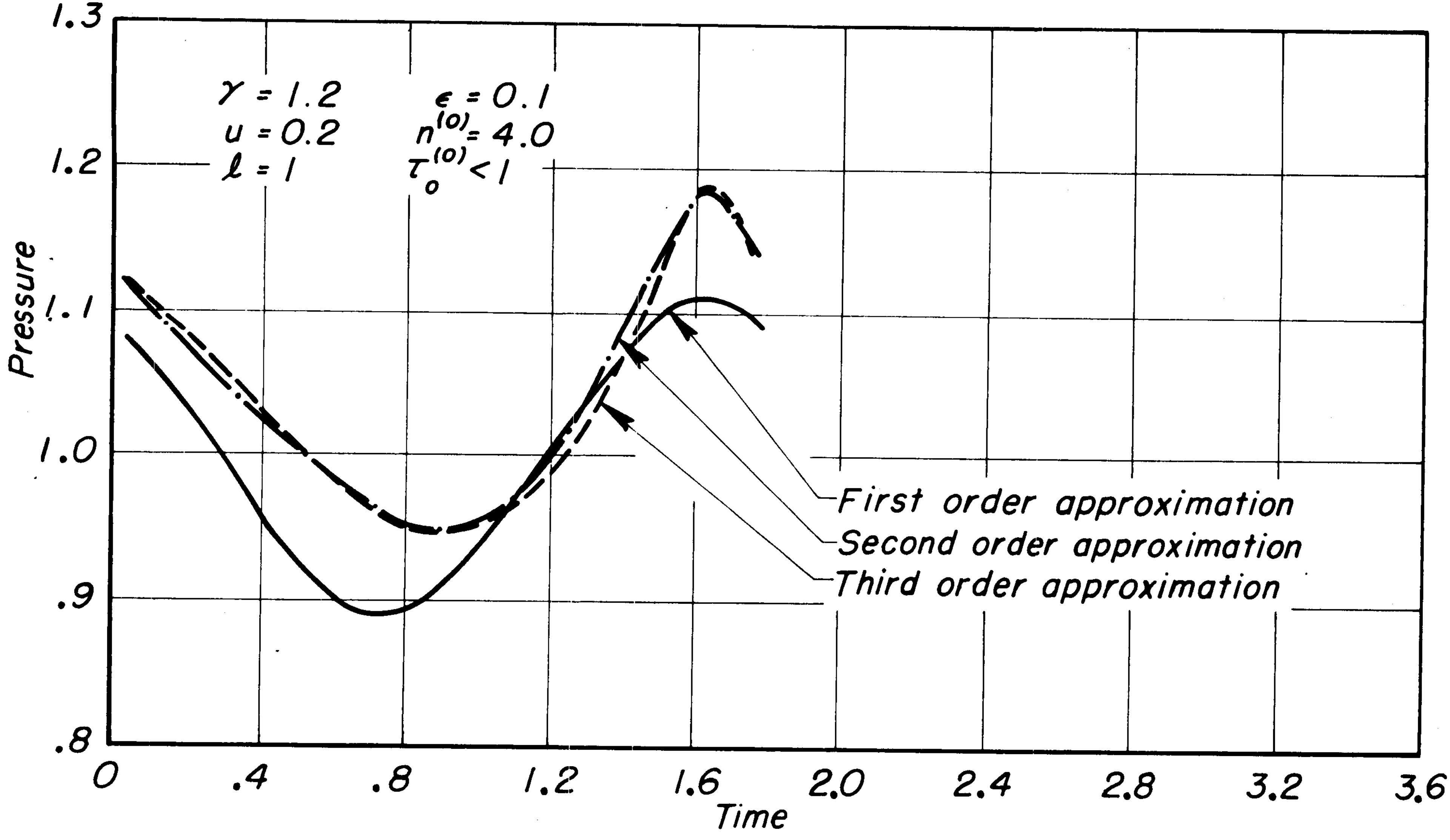


Figure 22

Pressure wave form at injector with below-resonant oscillation

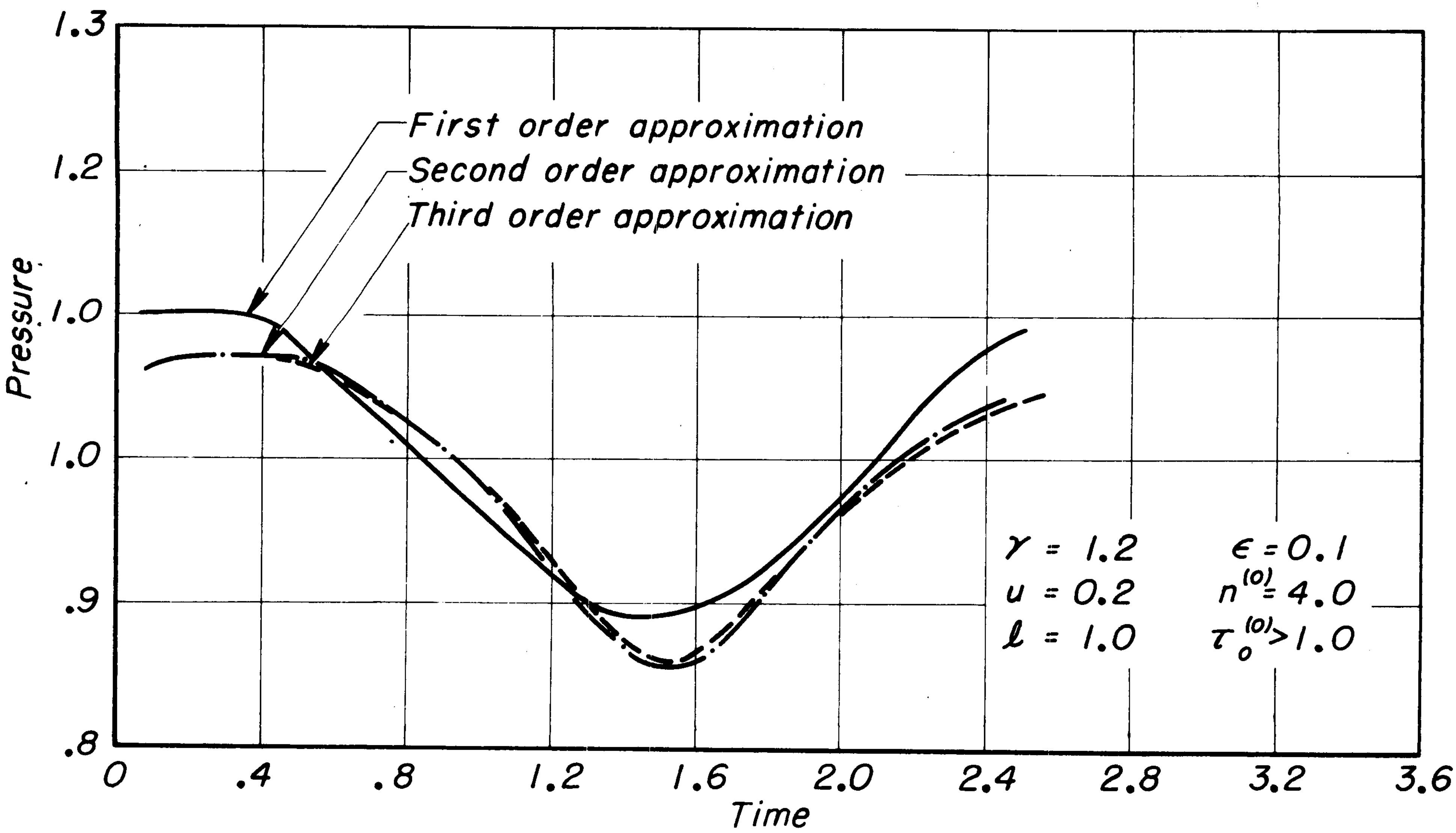
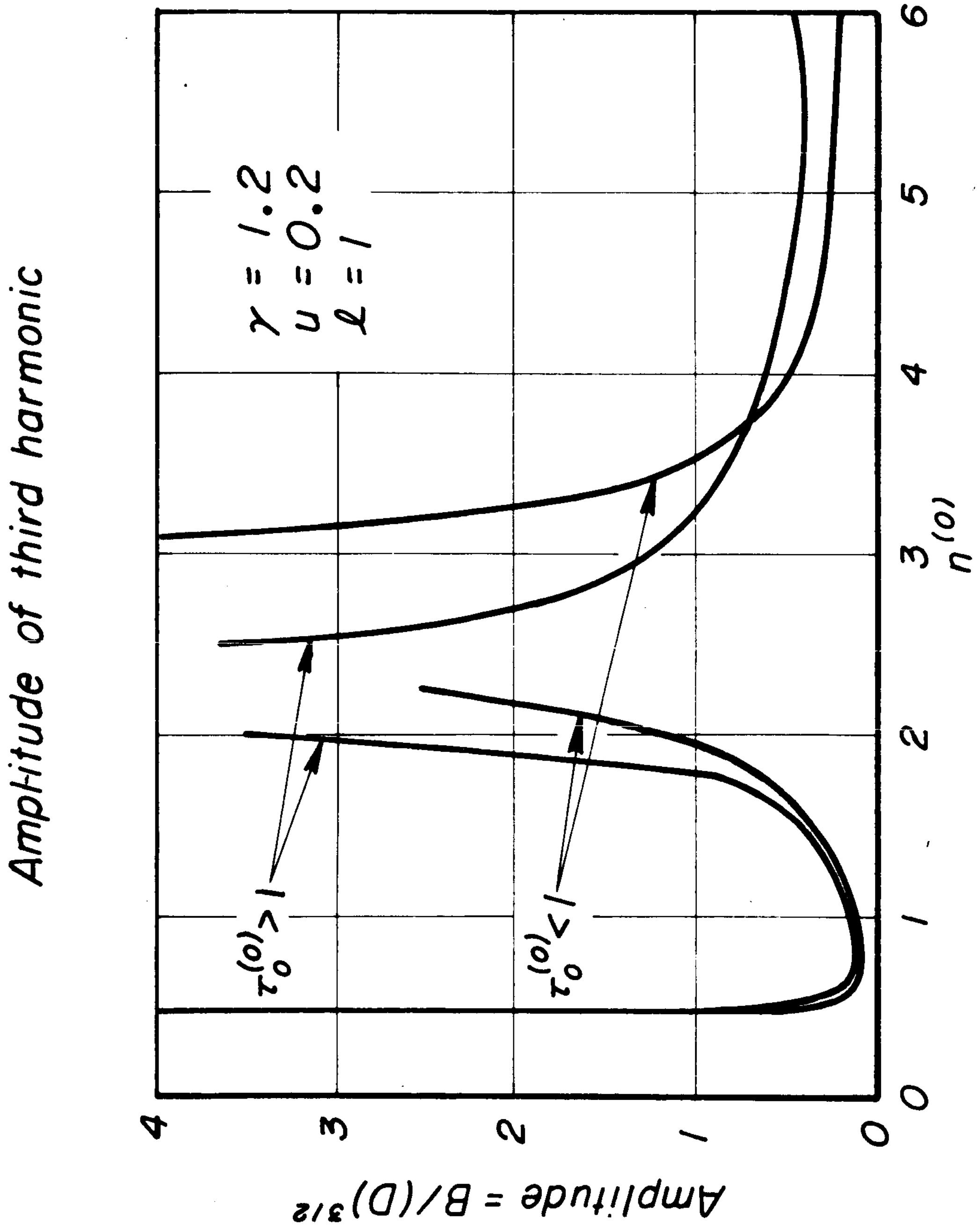


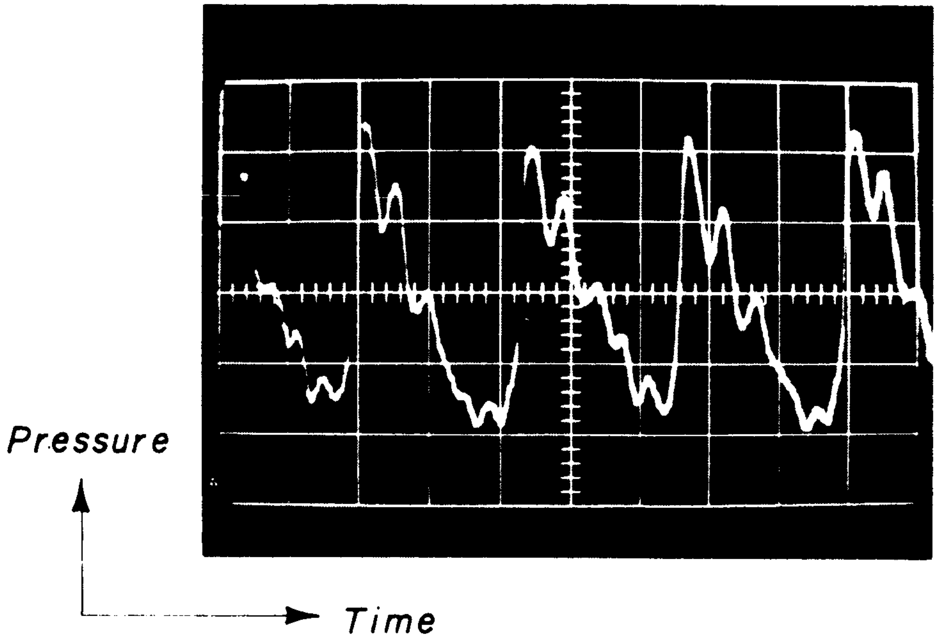
Figure 23



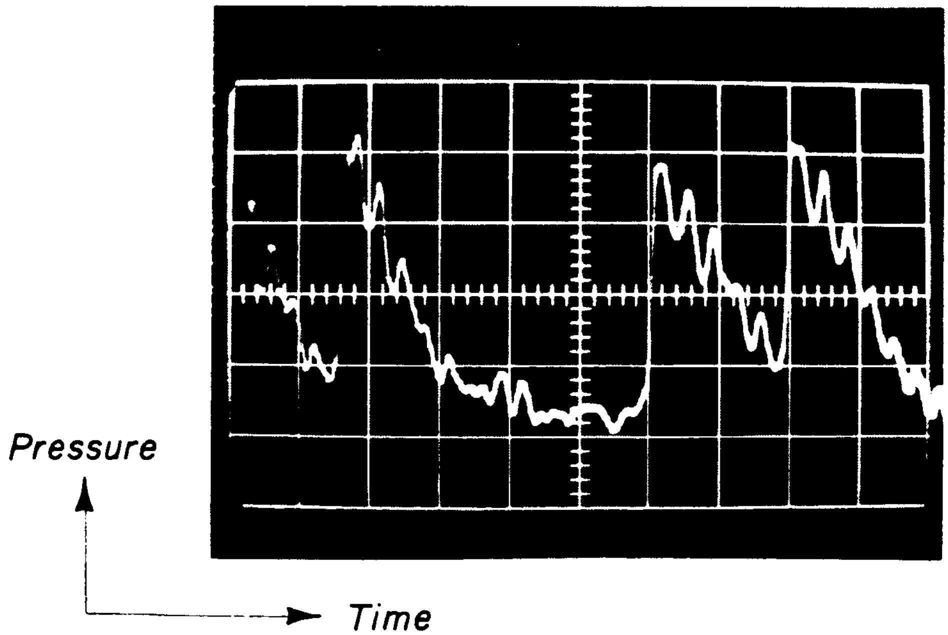
Amplitude of third harmonic
γ = 1.2
u = 0.2
ℓ = 1
τ₀^(0) > 1
τ₀^(0) < 1
Amplitude = B/(D)^{3/2}
n^(0)
0
1
2
3
4
5
6

Figure 24

Experimental results from Princeton gas rocket: chamber pressure=100 psi, equivalence ratio =2.7



Motor length=15"
$\frac{\Delta P}{P} = 0.144$



Motor length=30"
$\frac{\Delta P}{P} = 0.115$

Figure 25

Neutral stability curves

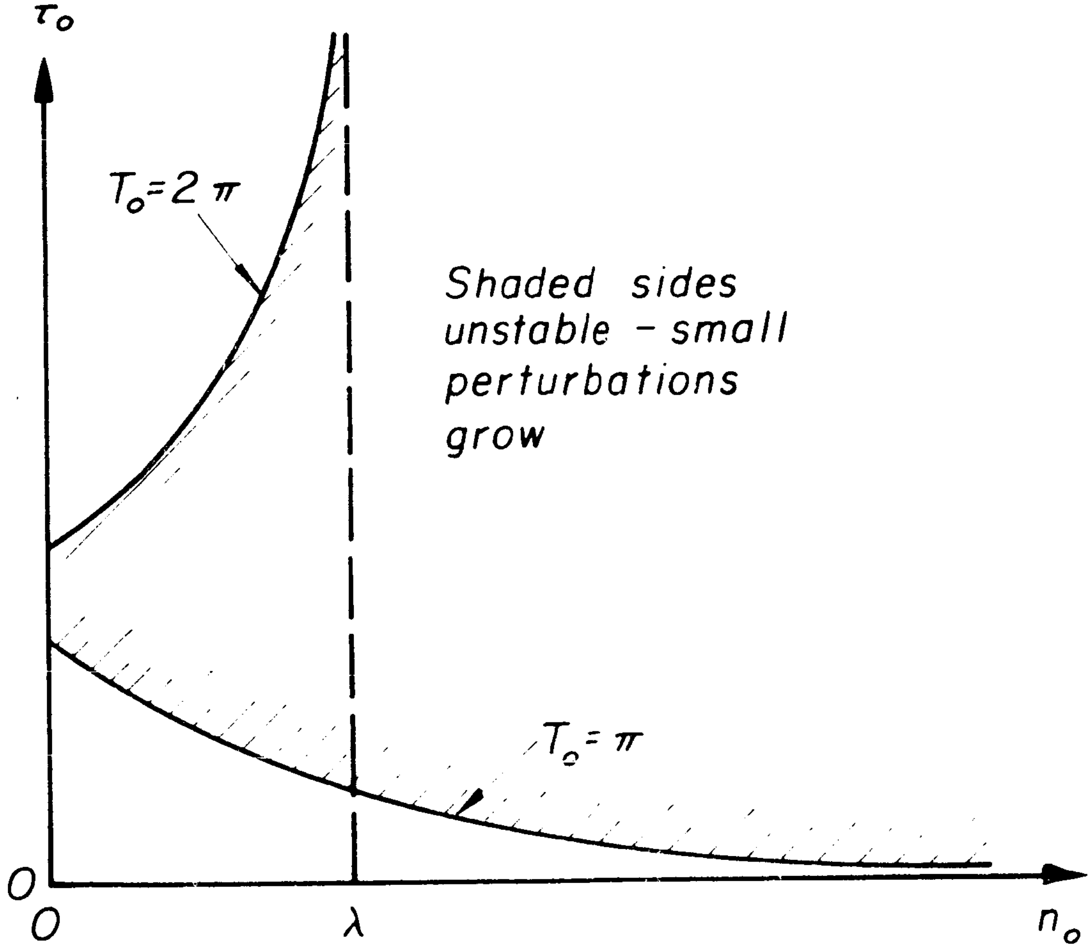


Figure A1

Effect of displacement from neutral line: $m=3$

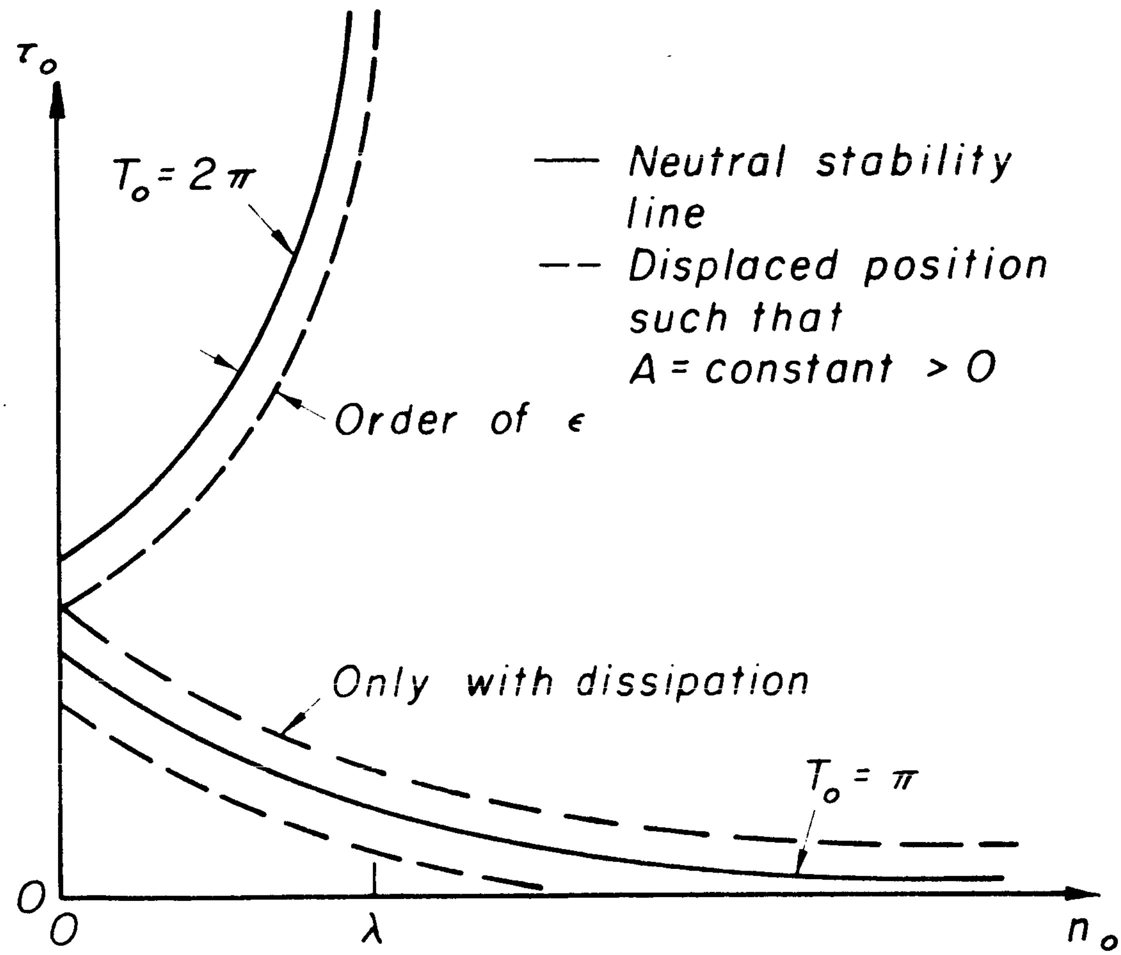


Figure A2

Effect of displacement from neutral line : m=2

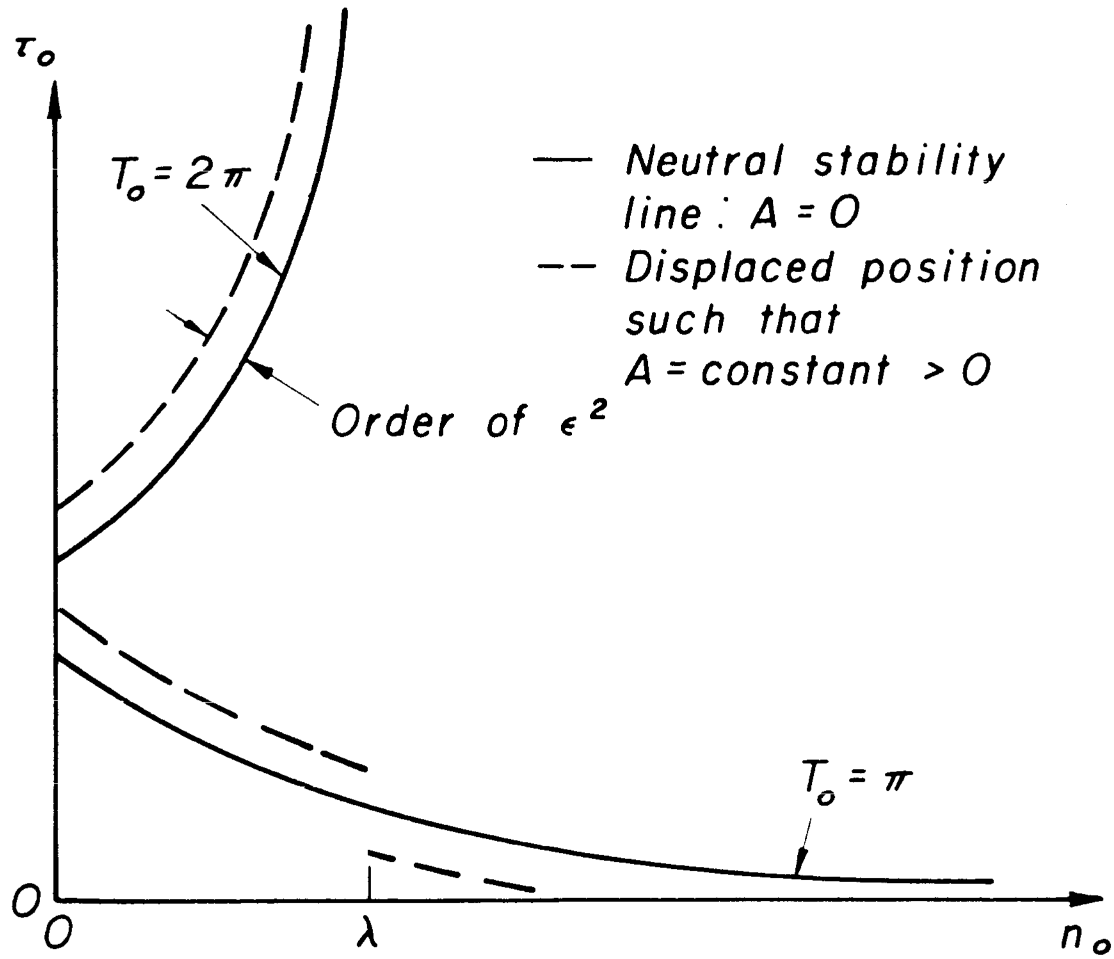


Figure A3

Solutions to differential equations

Case I: $C>0$

Figure D-1a

Subcase IA: $\lambda c + r > 0$, $\lambda > 0$

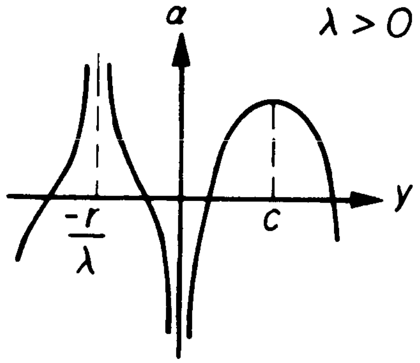


Figure D-1b

Subcase IB: $\lambda c + r > 0$, $\lambda = 0$

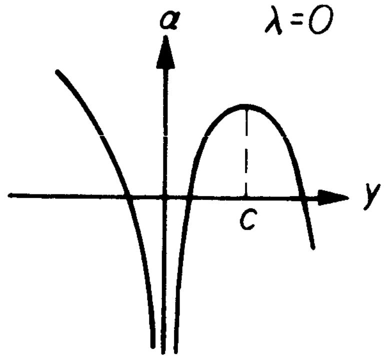


Figure D-1c

Subcase IC: $\lambda c + r > 0$, $\lambda < 0$

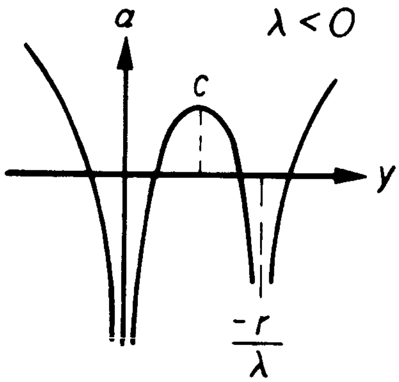


Figure D-1d

Subcase ID: $\lambda c + r < 0$

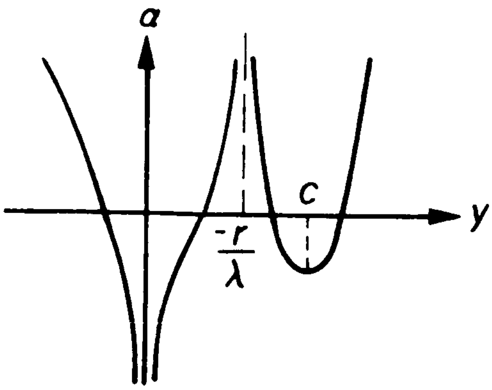


Figure D-1e

Subcase IE: $\lambda c + r = 0$

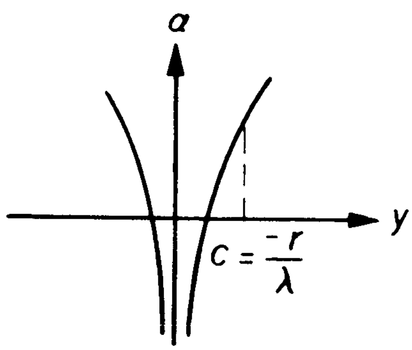


Figure D-1

Solutions to differential equations

Case II : $C < 0$

Figure D-2a

Subcase IIA : $\lambda c + r > 0$
$\lambda > 0$

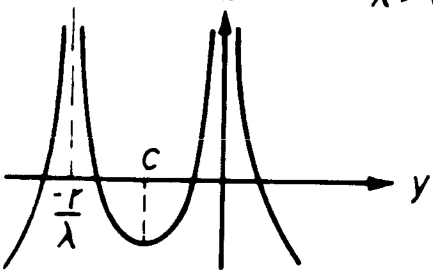


Figure D-2b

Subcase IIB : $\lambda c + r > 0$
$\lambda = 0$

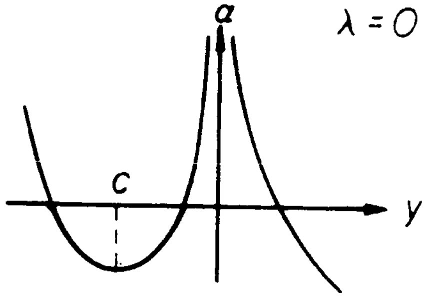


Figure D-2c

Subcase IIC : $\lambda c + r > 0$
$\lambda < 0$

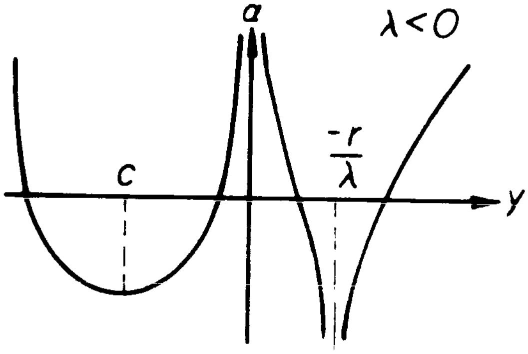


Figure D-2d

Subcase IID : $\lambda c + r < 0$

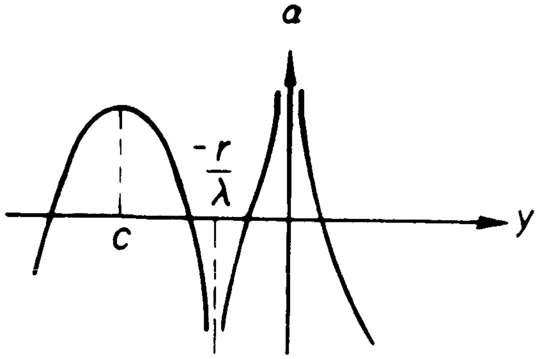


Figure D-2e

Subcase II E : $\lambda c + r = 0$

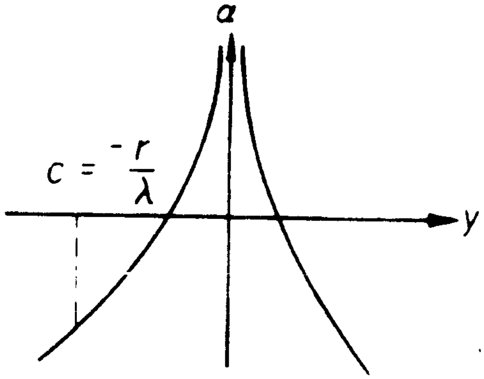


Figure D-2

Solutions to differential equations
Case III : $c = 0$

Figure D-3a
Subcase III A : $\lambda > 0$

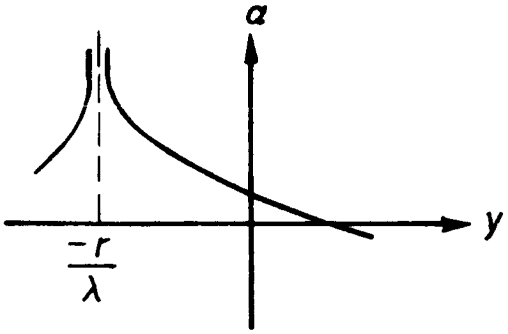


Figure D-3b
Subcase III B : $\lambda = 0$

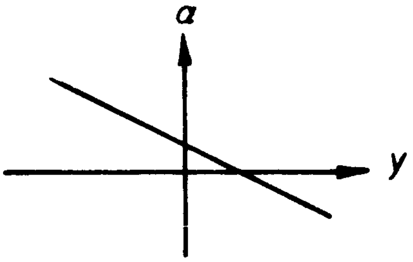
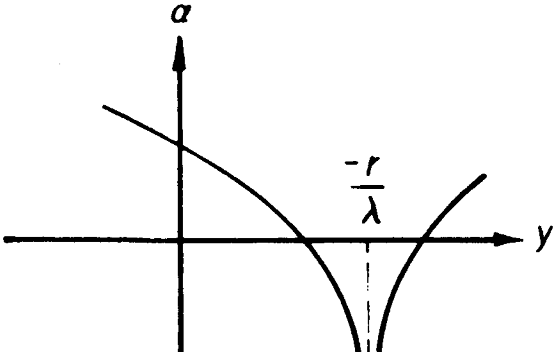


Figure D-3c
Subcase III C : $\lambda < 0$



JPR 2196

Figure D-3

APPENDIX A

NONLINEAR MECHANICAL ANALOGY

The following ordinary differential-difference equation is considered

$$\frac{d^2x}{dt^2} + \mu\frac{dx}{dt} + \lambda x - n x_\tau + \epsilon x^m = 0$$

where $\mu, \epsilon, \lambda, n,$ and τ are positive or zero only and m is some positive integer greater than unity. x_τ represents $x(t-\tau)$ and implies a delayed action with time lag τ. μ and ϵ are considered small and of the same order of magnitude. Therefore, both the nonlinear effect and the dissipative effect are small.

Periodic solutions only are desired. If ω is the angular frequency of the solution, it is convenient to make the transformations:

$$\rho = \omega t \qquad T = \omega\tau$$

The differential-difference equation becomes

$$\omega^2\frac{d^2x}{d\rho^2} + \mu\omega\frac{dx}{d\rho} + \lambda x - n x_T + \epsilon x^m = 0 \tag{A-1}$$

ρ is now the independent variable with ω unknown. The transformation is convenient because now the period of the oscillation is known (equals 2π). The transformation back to the time dimension, however, cannot be performed until ω is determined.

A perturbation technique is used to solve the equation with ϵ taken as the perturbation parameter. The perturbation series for x is written as follows:

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$$

n, ω, and T (or τ) are "eigenvalues" and, therefore, will also be written in series form (Note they do not satisfy the strictest mathematical definition of eigenvalues but they are characteristic values of the solutions). The series are written in the following manner:

$$n = n_o + \epsilon n_1 + \epsilon^2 n_2 + \text{----}$$
$$\omega = \omega_o + \epsilon\omega_1 + \epsilon^2 \omega_2 + \text{----}$$
$$T = T_o + \epsilon T_1 + \epsilon^2 T_2 + \text{----}$$
$$\tau = \tau_o + \epsilon\tau_1 + \epsilon^2 \tau_2 + \text{----}$$

The eigenvalue perturbations are included since the values of these parameters depend on the amplitude of the periodic oscillation. Conversely, the amplitude of the oscillation varies with these parameters.

Since μ and ϵ are of the same order, it is convenient to say $\mu = k\epsilon$ where $k = O[1]$. Now, the series substitutions are made into Equation (A-1). The equation is separated according to powers of ϵ and the zero order equation becomes:

$$\omega_o^2 \frac{d^2 x_o}{d\rho^2} + \lambda x_o - n_o x_{o_{T_o}} = 0$$

A periodic solution is of the form $x_o = A \cos\rho$ (where A is the undetermined amplitude). The characteristic equation is therefore found to be the following:

$$-\omega_o^2 + \lambda - n_o e^{-iT_o} = 0$$

This equation determines the neutral stability line in ω vs. n space (or in τ vs. n space). Since it is a complex relation, it may be rewritten as two equations:

$$-n_o \sin T_o = 0$$
$$-\omega_o^2 + \lambda - n_o \cos T_o = 0$$

Actually, two curves result from these equations and they are given by the following relations:

$$\omega_o^2 = \lambda + n_o \quad \text{when } T_o = \omega_o \tau_o = \pi,\ 3\pi,\ \text{etc.}$$
$$\omega_o^2 = \lambda - n_o \quad \text{when } T_o = \omega_o \tau_o = 0,\ 2\pi,\ \text{etc.} \qquad \text{(A-2)}$$

Only the π and 2π cases are considered here. So, in τ, n space, the following hyperbolae result:

$$\left(\frac{\tau_o}{\pi}\right)^2 (\lambda + n_o) = 1 \quad \text{when } T_o = \pi$$
$$\left(\frac{\tau_o}{2\pi}\right)^2 (\lambda - n_o) = 1 \quad \text{when } T_o = 2\pi$$

These curves are shown in Figure A1. Note that only positive values of τ, n, and λ are considered. A stability analysis of Equation (1) shows that small perturbations grow for values of τ and n to the right (shaded side) of each curve. Small perturbation decay on the other side of each of these neutral stability curves.

Equations (A-2) yield interesting information about the solution. Whenever no time-lag effect exists (n = 0), the resonant frequency of the harmonic oscillator is obtained. Otherwise, the time lag is most important in determining the frequency. In fact, in the cases considered, the ratios of the period of oscillation to the time-lag are simply 2 and 1. Whenever the time-lag effect is most significant ($n > \lambda$), only one frequency is possible. However, whenever the time-lag effect is least significant, ($\lambda > n$), two frequencies are possible.

At these neutral stability lines, the amplitude is zero. Although at a small distance from these curves finite-amplitude oscillations may exist. The rest of the analysis deals with periodic solutions of finite amplitude for τ, n points close to the τ_o vs. n_o curves.

Now it is necessary to investigate the higher order equations. The first order equation is:

$$\omega_o^2 \frac{d^2 x_1}{d\rho^2} + \lambda x_1 - n_o \; x_{1_{T_o}} + 2\,\omega_o\,\omega_1 \frac{d^2 x_o}{d\rho^2} - n_1 \; x_{o_{T_o}}$$
$$+ n_o \left.\frac{dx_o}{d\rho}\right|_{T_o} T_1 + k\omega_o \; \frac{dx_o}{d\rho} + x_o^{\;m} = 0 \qquad \text{(A-3)}$$

The first case to be solved has the exponent $m = 3$. The substitution $x_o = A \cos\rho$ is made into Equation (3). It is seen that x_1 has the particular solution $x_1 = c_1 \cos 3\rho$, where c_1 is an undetermined constant. Substituting for x_1 and separating coefficients of $\cos 3\rho$, $\cos\rho$, and $\sin\rho$ in Equation (A-3), the following three relations result:

$$\left[-9\omega_o^2 + \lambda \pm n_o\right] c_1 + \frac{A^3}{8} = 0$$

$$\frac{A}{2}\left[-2\,\omega_o\,\omega_1 \pm n_1\right] + \frac{3A^3}{8} = 0 \qquad \text{(A-4)}$$

$$A\left[\mp n_o T_1 + k\omega_o\right] = 0$$

The upper sign is understood when $T_o = \pi$ and the lower sign is understood when $T_o = 2\pi$.

If use is made of Equations (A-2) and note is taken that $T_1 = \omega_o \tau_1 + \omega_1 \tau_o$, Equations (A-4) yield the results:

$$c_1 = \frac{A^3}{64\omega_o^2}$$

$$\pm \frac{k\omega_o^2}{n_o \tau_o} - \frac{\omega_o}{\tau_o}\;\tau_1 \mp \frac{n_1}{2} = \frac{3A^3}{8}$$

$$\pm\, n_o \; T_1 = k\,\omega_o \,.$$

Note that c_1 is real and positive, so that the third harmonic is added with zero phase. c_1 is determined as a function of A which depends on the displacement from the neutral stability line. Therefore, given n_1 and τ_1, A and c_1 may be determined. In the absence of dissipative effects ($k = 0$), the ratio of the period to the time-lag remains unchanged.

For the case $T_o = \pi$, periodic oscillations are permitted for τ, n points both above and below the neutral line. However, oscillations at points above the neutral line are possible only for sufficiently large dissipative effects. Oscillations are possible below the neutral line with, or without, dissipation. For the case $T_o = 2\pi$, periodic solutions are only possible at τ, n points to the right of the neutral line. (See Figure A2). The interesting result is that, for the case $T_o = \pi$, finite oscillations are found at points where a linearized analysis would predict a stable situation. Therefore, it is possible for nonlinear effects to produce an extension of the instability region.

The second case to be examined has the value $m = 2$. In this case, no dissipative effects ($k = 0$) are considered. Again, the substitution $x_o = A \cos \rho$ is made into Equation (A-3). The particular solution is $x_1 = c_2 \cos 2\rho + c_3$ where c_2 and c_3 are undetermined constants. Upon substitution for x_1 into Equation (A-3) and separation of the coefficients of $\cos 2\rho$, 1, $\cos \rho_1$ and $\sin \rho$, the following four relations are obtained:

$$\left[-4\omega_0^2+\lambda-n_0\right]c_2+\frac{A^2}{4}=0$$

$$(\lambda-n_0)\,c_3+\frac{A^2}{2}=0$$

$$A\left[-2\,\omega_0\,\omega_1\pm n_1\right]=0$$

$$A\left[\mp n_0\;T_1\right]=0 \qquad \text{(A-5)}$$

Through use of Equations (A-2) and of the relationship $T_1 = \omega_0 \tau_1 + \omega_1 \tau_0$, the following is obtained from Equation (A-5):

$$c_2=\frac{A^2}{16(\lambda\pm n_0)-4(\lambda-n_0)}$$

$$c_3=\frac{A^2}{2(n_0-\lambda)}$$

$$\frac{-\omega_0^2\,\tau_1}{\tau_0}\mp\frac{n_1}{2}=0 \qquad \text{(A-6)}$$

$$T_1=0$$

c_2 is positive and real, so that the second harmonic is added without a phase. c_3 is positive whenever $n_0 > \lambda$ and negative whenever $n_0 < \lambda$. In each case ($T_0 = \pi$ or $T_0 = 2\pi$), the relation between τ_1 and n_1 implies a displacement tangent to the τ_0 vs. n_0 curve rather than away from the curve. This is a trivial displacement in τ vs. n space and no generality is lost by saying $\tau_1 = n_1 = 0$. If there exists a nontrivial displacement which will yield finite amplitudes, the perturbations in τ and n must be of higher order.

For the $m = 2$ case, it is necessary to solve the second order equation in order to determine the eigenvalue perturbations. This equation is written as follows:

$$\omega_0^2 \frac{d^2 x_2}{d\rho^2} + \lambda x_2 - n_0 \, x_{2\,\tau_0} + 2\,\omega_0\,\omega_2 \frac{d^2 x_0}{d\rho^2} -$$
$$n_2 \, x_{0\,\tau_0} + n_0 \left.\frac{d x_0}{d\rho}\right|_{\tau_0} \tau_2 + 2\,x_0\,x_1 = 0 \tag{A-7}$$

Upon substitution for x_0 and x_1 in Equation (A-7), the particular solution $x_2 = c_4 \cos 3\rho$ is seen (where c_4 is an undetermined constant). Now, after substitution of the solution for x_2 in Equation (A-7) and separation of the coefficients of $\cos 2\rho$ $\sin\rho$, and $\cos\rho$, the following results are found:

$$\left[-9\,\omega_0^2 + \lambda \pm n_0\right] c_4 + A\,c_2 = 0$$
$$A\left[\mp \tau_2\right] = 0$$
$$A\left[-\omega_0\,\omega_2 \pm \frac{n_2}{2} + c_2 + c_3\right] = 0 \tag{A-8}$$

It is seen that $T_2 = \omega_0\,\tau_2 + \omega_2\,\tau_0$ for this case. After this substitution, Equations (A-2), (A-6), and (A-8) yield:

$$c_4 = \frac{A^3}{32(\lambda \pm n_0)^2\left[4(\lambda \pm n_0) - (\lambda - n_0)\right]}$$
$$T_2 = 0$$
$$-\omega_0^2 \frac{\tau_2}{\tau_0} \mp \frac{n_2}{2} = A^2\left[\frac{1}{2(n_0 - \lambda)} + \frac{1}{16(\lambda \pm n_0) - 4(\lambda - n_0)}\right]$$

c_4 is real and positive so that the third harmonic is also added without any phase.

The final relation determines the perturbations in τ and n necessary to obtain a given amplitude A or, conversely, the amplitude depends on the displacement from the neutral line. As is shown in Figure A3, the unstable region (in γ, n space) is extended by nonlinear effects for the case $T_o = 2\pi$. For the other case ($T_o = \pi$), the unstable region is extended whenever $n_o > \lambda$. However, this is not true whenever $\lambda > n_o$. There, periodic solutions are obtained in the unstable region only. In the cases where the instability region is extended, it has not been possible to find periodic solutions within the linearly unstable region (region where infinitesimally small perturbations grow to a finite magnitude). In order that these periodic solutions be found, certain nonlinear dissipative effects must be present.

Note, in the $m = 2$ case, an eigenvalue perturbation of $O(\epsilon^2)$ produces a finite amplitude while a perturbation of $O(\epsilon)$ is necessary in the $m = 3$ case. This implies that finite amplitudes occur much closer to the neutral line in the $m = 2$ case than in the $m = 3$ case.

The important conclusions are that: (1) off-resonant oscillations are possible for both linear and nonlinear phenomena, (2) the region of instability may be extended due to nonlinear effects, (3) the amplitude of the oscillation depends on the displacement from the neutral line in some eigenvalue plot, and (4) the eigenvalue perturbations may be of higher order.

APPENDIX B

INVESTIGATION OF COMBUSTION ZONE DYNAMICS

Consider a one-dimensional combustion zone near the injector of a premixed gas rocket (or the receding surface of an end-burning solid propellant rocket). Assuming the gas is calorically perfect, the governing equations*are:

Equation of State: $P = \rho T$

Continuity Equation: $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$

Energy Equation: $\frac{\gamma}{\gamma - 1} \rho \left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x}\right) = \frac{\partial P}{\partial t} + r + \frac{\partial}{\partial x}\left[(\epsilon_h + \lambda)\frac{\partial T}{\partial x}\right]$

All thermodynamic variables are nondimensionalized with respect to their steady-state values at the point of completed combustion; Velocity u is nondimensionalized with respect to the steady-state speed of sound at this point of completed combustion; time t with respect to chamber length divided by speed of sound; space dimension x with respect to chamber length; energy release rate per unit volume r with respect to steady-state pressure times speed of sound divided by chamber length. Thermal conductivity and turbulent exchange coefficient are nondimensionalized with respect to pressure times speed of sound times chamber length divided by temperature. This same non-dimensionalization scheme is used in the analysis of the chamber gas dynamics.

The momentum equation is replaced by the assumption that the pressure gradient is zero throughout the zone. Actually, the pressure gradient is of the order of the Mach number and is taken as negligible in small Mach number chambers. These

*The species equation is omitted since r is assumed to be a function of thermodynamic conditions only.

equations may be combined to yield the equation:

$$P\frac{\partial u}{\partial x} = -\frac{1}{\gamma}\frac{dP}{dt} + \frac{\gamma-1}{\gamma}r + \frac{\gamma-1}{\gamma}\frac{\partial}{\partial x}\left[(\epsilon_h+\lambda)\frac{\partial T}{\partial x}\right] \quad \text{(B-1)}$$

Equation (B-1) can be integrated with respect to x from x =0 (injection surface) to x = ℓ which is some position at which the reaction may be assumed complete. Noting that ϵ_h = 0 at x = 0 and $\frac{\partial T}{\partial x} = 0$ at $x = \ell$, the following is obtained:

$$P(u_{x=\ell} - u_{x=0}) = -\frac{dP}{dt}\frac{\ell}{\gamma} + \frac{\gamma-1}{\gamma}\int_0^\ell r\,dx - \frac{\gamma-1}{\gamma}\left[\lambda\frac{\partial T}{\partial x}\right]_{x=0}$$

which may be rewritten as

$$pu_{x=\ell} = -\frac{dP}{dt}\frac{\ell}{\gamma} + \frac{\gamma-1}{\gamma}\int_0^\ell r\,dx + \left[\dot{m}T - \frac{\gamma-1}{\gamma}\lambda\frac{\partial T}{\partial x}\right]_{x=0} \quad \text{(B-2)}$$

Equation (B-2) may be simplified by assuming that the perturbation of heat transferred to the injector surface (or solid propellant surface) and the perturbation of convected energy through that surface are negligibly small compared to the perturbation of energy of combustion. Note that this allows no change in the mean burning rate due to the oscillation. Letting zero subscript denote steady-state values, the steady-state equation states:

$$u_{0,x=\ell} = \frac{\gamma-1}{\gamma}\int_0^\ell r_0\,dx + \left[\dot{m}T_0 - \frac{\gamma-1}{\gamma}\lambda\frac{dT_0}{dx}\right] \quad \text{(B-3)}$$

Allowing primes to denote perturbation quantities, Equation

(B-3) is subtracted from (B-2) to yield

$$(pu)'_{x=\ell} = -\frac{dP}{dt}\frac{\ell}{\gamma} + \frac{\gamma-1}{\gamma}\int_0^{\ell} r'\, dx \qquad \text{(B-4)}$$

$(pu)'_{x=\ell}$ represents the energy put into the oscillation of the chamber gases. According to equation (B-4), this is large when the reaction zone is spread out in distance (large ℓ). In other words, the rocket becomes more unstable as the reaction zone is lengthened. It is stressed, therefore, that according to this model, lower chamber pressures would cause a more unstable situation.

The energy release rate is a function of temperature and pressure and may be expanded in a double Taylor series about the steady state values. So that

$$r = r_0 \Big\{ 1 + \left(\frac{1}{r}\frac{\partial r}{\partial p}\right)_0 (p-1) + \left(\frac{1}{r}\frac{\partial r}{\partial T}\right)_0 (T-1) +$$
$$\frac{1}{2}\left(\frac{1}{r}\frac{\partial^2 r}{\partial p^2}\right)_0 (p-1)^2 + \left(\frac{1}{r}\frac{\partial^2 r}{\partial p\,\partial T}\right)_0 (p-1)(T-1) +$$
$$\frac{1}{2}\left(\frac{1}{r}\frac{\partial^2 r}{\partial T^2}\right)_0 (T-1)^2 + ---- \Big\} \qquad \text{(B-5)}$$

Substituting Equation (B-5) into Equation (B-2) and making use of Equation (B-3), the result is:

$$\left(\frac{u-u_0}{u_0}\right)_{x=\ell} = -(p-1) + \frac{\int_0^{\ell} r_0 \left[\left(\frac{1}{r}\frac{\partial r}{\partial p}\right)(p-1) + \left(\frac{1}{r}\frac{\partial r}{\partial T}\right)_0 (T-1)\right] dx}{\int_0^{\ell} r_0\, dx} -$$

$$-(p-1)\left(\frac{u-u_0}{u_0}\right)_{x=\ell} + \frac{\int_0^{\ell} r_0 \left[\frac{1}{2}\left(\frac{1}{r}\frac{\partial^2 r}{\partial p^2}\right)_0 (p-1)^2 + \left(\frac{1}{r}\frac{\partial^2 r}{\partial p\,\partial T}\right)_0 (T-1)(p-1) + \frac{1}{2}\left(\frac{1}{r}\frac{\partial^2 r}{\partial T^2}\right)_0 (T-1)^2\right] dx}{}$$

$$-\frac{dp}{dt}\frac{\ell}{\gamma} + ----$$

Now, it shall be assumed that the reaction zone is so thin that the time derivative term is negligible. Also, the integrals shall be replaced by using mean values of r, its derivatives, and T in the combustion zone. This yields,

$$\left(\frac{u-u_o}{u_o}\right)_{x=\ell} = -(p-1) + \left[\left(\frac{1}{r}\frac{\partial r}{\partial p}\right)_o (p-1) + \left(\frac{1}{r}\frac{\partial r}{\partial T}\right)_o (T-1)\right]_M$$

$$-(p-1)\left(\frac{u-u_o}{u_o}\right)_{x=\ell} + \left[\frac{1}{2}\left(\frac{1}{r}\frac{\partial^2 r}{\partial p^2}\right)_o (p-1)^2 + \left(\frac{1}{r}\frac{\partial^2 r}{\partial p \partial T}\right)_o (T-1)(p-1) + \right.$$

$$\left.\frac{1}{2}\left(\frac{1}{r}\frac{\partial^2 r}{\partial T^2}\right)_o (T-1)^2\right]_M + ---- \qquad \text{(B-6)}$$

where the subscript M denotes a mean value. Note that r_o possesses a maximum somewhere in the combustion zone and that the mean values are approximately the local values at some point near the maximum point.

Now, of course, the temperature perturbation (T-1) must be related to the pressure perturbation (p-1). The relationship between these two perturbations differs from the isentropic relationship due to two effects; release of energy and diffusion within the reaction zone. Diffusion effects are negligible when the diffusion relaxation time is much longer than the period of oscillation; however, the chemical relaxation time is almost always much shorter than the period of the oscillation. Rather than attempting a solution of the nonlinear, parabolic equation which would give the correct relationship, the isentropic relationship shall be assumed. Therefore, diffusion relaxation time is assumed very long and variations in entropy production due to combustion are deliberately neglected. Note that this isentropic relationship will apply through second order in wave amplitude even when a shock wave moves through the combustion zone. Under these assumptions there will be no time lag or delay in the system.

It is convenient to related (p-1) and (T-1) to (a-1) through the isentropic relationship obtaining in series form:

$$(p-1) = \frac{2\gamma}{\gamma-1}(a-1) + \frac{\gamma(\gamma+1)}{(\gamma-1)^2}(a-1)^2 + \cdots$$

$$(T-1) = 2(a-1) + (a-1)^2 \tag{B-7}$$

Substituting Equations (B-7) into Equation (B-6), the following result is obtained:

$$\left(\frac{u-u_o}{u_o}\right)_{x=\ell} = \left[2\frac{1}{r}\frac{\partial r}{\partial T} + \frac{2\gamma}{\gamma-1}\frac{1}{r}\frac{\partial r}{\partial p} - \frac{2\gamma}{\gamma-1}\right]_o (a-1) -$$

$$- \left(\frac{u-u_o}{u_o}\right)_{x=\ell}\frac{2\gamma}{\gamma-1}(a-1) +$$

$$+ \left[\frac{\gamma(\gamma+1)}{(\gamma-1)^2}\frac{1}{r}\frac{\partial r}{\partial p} + \frac{1}{r}\frac{\partial r}{\partial T} - \frac{\gamma(\gamma+1)}{(\gamma-1)^2} + 2\left(\frac{\gamma}{\gamma-1}\right)^2\frac{1}{r}\frac{\partial^2 r}{\partial p^2} + \right.$$

$$\left. + \frac{4\gamma}{\gamma-1}\frac{1}{r}\frac{\partial^2 r}{\partial p\,\partial T} + 2\frac{1}{r}\frac{\partial^2 r}{\partial T^2}\right](a-1)^2 + \cdots$$

It is understood that mean steady-state values are used for the coefficients of (a-1), $(a-1)^2$, etc. These are the same mean values as appeared in Equation (B-6).

Now, using the method of successive approximations to substitute for $\left(\frac{u-u_o}{u_o}\right)_{x=\ell}$ on the right-hand side of this equation, the final relation becomes:

$$\left(\frac{u-u_o}{u_o}\right)_{x=l} = \left[2\frac{1}{r}\frac{\partial r}{\partial T} + \frac{2\gamma}{\gamma-1}\frac{1}{r}\frac{\partial r}{\partial p} - \frac{2\gamma}{\gamma-1}\right]_o (a-1)$$

$$-\frac{2\gamma}{\gamma-1}\left[2\frac{1}{r}\frac{\partial r}{\partial T} + \frac{2\gamma}{\gamma-1}\frac{1}{r}\frac{\partial r}{\partial p} - \frac{2\gamma}{\gamma-1}\right]_o (a-1)^2 +$$

$$+\left[\frac{\gamma(\gamma+1)}{(\gamma-1)^2}\frac{1}{r}\frac{\partial r}{\partial p} + \frac{1}{r}\frac{\partial r}{\partial T} + 2\left(\frac{\gamma}{\gamma-1}\right)^2\frac{1}{r}\frac{\partial^2 r}{\partial p^2} + \frac{4\gamma}{\gamma-1}\frac{1}{r}\frac{\partial^2 r}{\partial p \partial T} +\right.$$

$$\left. + 2\frac{1}{r}\frac{\partial^2 r}{\partial T^2} - \frac{\gamma(\gamma+1)}{(\gamma-1)^2}\right]_o (a-1)^2 + \dots \qquad \text{(B-8)}$$

Equation (B-8) is a nonlinear relation which states in an approximate manner the perturbation in the outflow velocity from the combustion zone due to a perturbation in the mean thermodynamic conditions within the combustion zone. Therefore, this is the governing relation for the feedback of energy to the instability.

By comparison with Equation 9 in Chapter II, it is seen that for this type of combustion process

$$\omega \equiv \left[2\frac{1}{r}\frac{\partial r}{\partial T} + \frac{2\gamma}{\gamma-1}\frac{1}{r}\frac{\partial r}{\partial p} - \frac{2\gamma}{\gamma-1}\right]_o$$

$$\delta \equiv -\frac{2\gamma\omega}{\gamma-1} + \left[\frac{\gamma(\gamma+1)}{(\gamma-1)^2}\frac{1}{r}\frac{\partial r}{\partial p} + \frac{1}{r}\frac{\partial r}{\partial T} + 2\left(\frac{\gamma}{\gamma-1}\right)^2\frac{1}{r}\frac{\partial^2 r}{\partial p^2} +\right.$$

$$\left. + \frac{4\gamma}{\gamma-1}\frac{1}{r}\frac{\partial^2 r}{\partial p \partial T} + 2\frac{1}{r}\frac{\partial^2 r}{\partial T^2}\right]_o - \frac{\gamma(\gamma+1)}{(\gamma-1)^2} \qquad \text{(B-9)}$$

APPENDIX C

SMALL PERTURBATION ANALYSIS

The stability of the steady-state operation is examined for the configuration as already defined in Chapter II; that is, a one-dimensional flow in a constant-area chamber with a zero-length nozzle and a combustion zone concentrated at the chamber end. Also, the perfect gas assumption is made.

The transformation to characteristic coordinates is made with the following numbering system for the coordinates: $\alpha = \beta$ at $x = 0$ (combustion zone), $\alpha = 1 + \beta$ at $x = 1$ (nozzle) and $\alpha = \beta = 0$ at the initial point, $x = t = 0$.

The equations of the system are given by Equation (2) Chapter II :

$$\frac{2}{\gamma-1}\frac{\partial a}{\partial \alpha} + \frac{\partial u}{\partial \alpha} = 0$$

$$\frac{2}{\gamma-1}\frac{\partial a}{\partial \beta} - \frac{\partial u}{\partial \beta} = 0$$

$$\frac{\partial x}{\partial \alpha} = (u + a)\frac{\partial t}{\partial \alpha}$$

$$\frac{\partial x}{\partial \beta} = (u - a)\frac{\partial t}{\partial \beta}$$

The usual nondimensional scheme is used: thermodynamic variables are nondimensionalized with respect to their steady-state variables and velocity is nondimensionalized with respect to steady-state speed of sound. Further, space dimension is nondimensionalized with respect to chamber length; time with respect to chamber length divided by the speed of sound.

For the purposes of this analysis, the first two equations are of main interest. That is, it is desired to know the time-wise behavior of a small perturbation of the flow properties. Nonlinear corrections in the coordinates are not desired. So, the first two equations only are investigated in order to determine the stability of the system under the influence of small perturbations.

These equations have the following solutions:

$$P(\beta) = \frac{2}{\gamma-1} a + u$$
$$Q(\alpha) = \frac{2}{\gamma-1} a - u$$

All flow properties are considered as a steady-state term plus a perturbation term:

$$P(\beta) = P_o(\beta) + P'(\beta)$$
$$Q(\alpha) = Q_o(\alpha) + Q'(\alpha)$$
$$u = u_o(\alpha,\beta) + u'(\alpha,\beta)$$
$$a = 1 + a'(\alpha,\beta)$$

where zero subscripts denote steady-state quantities and primed symbols indicate perturbation values.

The perturbation equations are found to be the following:

$$P'(\beta) = \frac{2}{\gamma-1} a' + u'$$
$$Q'(\alpha) = \frac{2}{\gamma-1} a' - u'$$

so that

$$u' = \frac{P'(\beta) - Q'(\alpha)}{2}$$

$$a' = \frac{\gamma-1}{4}\left[P'(\beta) + Q'(\alpha)\right]$$

(C-1)

The boundary condition at the combustion zone results from perturbing Equation (9) in Chapter II. It may be written as follows:

$$u' = \omega\, u_o\, a' \qquad \text{at} \qquad \alpha = \beta$$

Substitution from Equation (C-1) results in the relationship:

$$(1-\omega\nu)\; P'(\beta) = (1+\omega\nu)\; Q'(\beta)$$

(C-2)

where $\nu \equiv \frac{\gamma-1}{2}\, u_o$

Equation (C-2) indicates that a wave gains strength in reflection at the combustion zone ($P'/Q' > 1$). This gain is expected due to energy addition from the combustion process.

The boundary condition at the nozzle entrance is written:

$$u' = u_o\, a' \qquad \text{at} \qquad \alpha = 1+\beta$$

For this case, substitution from Equation (C-1) yields the relationship:

$$(1-\nu)\; P'(\beta) = (1+\nu)\; Q'(1+\beta)$$

(C-3)

Equation (C-3) indicates that a wave loses strength in reflection at the nozzle ($Q'/P' < 1$). This loss results from the convection of energy of oscillation out of the chamber through the nozzle. Note that changes in wave strength may occur only at the nozzle or combustion zone.

Now, Equations (C-2) and (C-3) may be combined to yield the following relationship:

$$\frac{P'(1+\beta)}{P'(\beta)} = \frac{Q'(1+\beta)}{Q'(\beta)} = \left(\frac{1-\gamma}{1-\omega\gamma}\right)\left(\frac{1+\omega\gamma}{1+\gamma}\right) \equiv \mu$$

Note that the natural period of oscillation equals unity (in characteristic coordinates). So, the value of μ indicates the change in amplitude of the disturbance over a period of time. Specifically, whenever $\mu > 1$, exponential growth of the small disturbance occurs and whenever $\mu < 1$, exponential decay occurs. In the special case $\mu = 1$, a neutral oscillation of the small disturbance is obtained.

The following is readily seen: $\omega > 1$ implies $\mu > 1$, $\omega = 1$ implies $\mu = 1$, and $\omega < 1$ implies $\mu < 1$. Therefore, if $\omega > 1$, the motor is unstable; i.e., small disturbances grow until inhibited by nonlinear effects. The growth is forced to occur because more energy is added to the oscillation at the combustion zone than is removed at the nozzle. Whenever, $\omega < 1$, the opposite is true resulting in a stable situation; i.e., more energy is removed at the nozzle than is added at the combustion zone. Of course, whenever $\omega = 1$, the energy addition and energy removal are in balance, explaining the neutral oscillation.

APPENDIX D

INVESTIGATION OF SOLUTIONS TO DIFFERENTIAL EQUATION OF CHAPTER II

In Chapter II the differential equation which governed the waveform of the oscillation was found to be of the type

$$\frac{dy}{d\alpha} = \frac{ry + \lambda y^2}{c - y}$$

where

$$c = \frac{1}{2}\left[y(1) + y(0)\right]$$

This differential equation is related to the combustion instability phenomenon by means of the following definitions:

$$y(\alpha) \equiv Q_1(\alpha)$$

$$r \equiv \frac{16}{\gamma+1} \frac{1}{(1-v^2)\left[\frac{1-u_0}{1+u_0}\frac{1+v}{1-v} + \frac{1+u_0}{1-u_0}\right]}$$

$$\lambda \equiv \frac{16 u_0 \delta}{(\gamma+1)(1+v)} \frac{\left[\left(\frac{\gamma-1}{2}\right)\frac{1}{1-v}\right]^2}{\left[\frac{1-u_0}{1+u_0}\frac{1+v}{1-v} + \frac{1+u_0}{1-u_0}\right]}$$

The value of r is always positive for practical situations. λ is usually positive, but may be negative. A continuous solution for $y(\alpha)$ is sought in the range $0 \leq \alpha \leq 1$. Since expansion shocks are not allowed, negative jumps in y (or Q_1) are ruled out. Positive jumps (which correspond to compression shocks) are allowed at the endpoints (0 and 1). Since c is defined as the average of the values of y at the two endpoints and the solution is continuous between the endpoints, there must exist some value of α (in the given range) such that $y = c$ at that value.

It is more convenient to look at the reciprocal equation:

$$\frac{d\alpha}{dy} = \frac{c-y}{ry+\lambda y^2} \qquad \text{(D-1)}$$

Note that if $\alpha(y)$ is a solution, $\alpha(y)$ plus any constant is also a solution. So, any curve of the solutions may be translated in the α-direction on an α vs. y plot in order to satisfy one condition on the solution.

The topological structure depends very much on the values of λ, r, and c. The case $c > 0$ and $\lambda c+r > 0$ is considered first. Here, a maximum of $\alpha(y)$ exists at $y = c$. It is also seen that $\frac{d\alpha}{dy}$ becomes infinite at $y = 0$ and $y = -\frac{r}{\lambda}$. There are three separate structures within this first case depending upon whether λ is greater than, equal to, or less than zero.

If $\lambda > 0$, the following additional information is given by the differential equation

$$y < -\frac{r}{\lambda}, \quad \frac{d\alpha}{dy} > 0$$

$$0 > y > -\frac{r}{\lambda}, \quad \frac{d\alpha}{dy} < 0$$

$$0 < y < c, \quad \frac{d\alpha}{dy} > 0$$

$$y > c, \quad \frac{d\alpha}{dy} < 0$$

Figure D-1a shows the above information in graphic form. The solutions are continuous within each of three adjacent regions. They cannot, however, be continued from one region to the next except, perhaps, at infinity.

In the subcase $\lambda = 0$, the differential equation indicates the following:

$$y < 0 \quad , \quad \frac{d\alpha}{dy} < 0$$

$$0 < y < c \quad , \quad \frac{d\alpha}{dy} > 0$$

$$y > c \quad , \quad \frac{d\alpha}{dy} < 0$$

Figure D-1b gives a sketch of the solution for this subcase. Note that there are two continuous regions with a discontinuity between them. The results for the subcase $\lambda < 0$ are obtained in a similar fashion and are shown in Figure D-1c. It is seen that they are similar to the other subcases. The only possibility here is that $c < -\frac{r}{\lambda}$ (since $c > 0, \lambda c+r > 0$, $r > 0$, and $\lambda < 0$).

Next, the case $c > 0$ and $\lambda c+r < 0$ is examined. The only possibilities here are $\lambda < 0$ and $c > -\frac{r}{\lambda}$. A minimum is found at $y = c$ and infinite slopes are indicated at $y = 0$ and $y = -\frac{r}{\lambda}$. In addition, the differential equation indicates that:

$$y < 0 \quad , \quad \frac{d\alpha}{dy} < 0$$

$$0 < y < \frac{r}{\lambda} \quad , \quad \frac{d\alpha}{dy} > 0$$

$$-\frac{r}{\lambda} < y < c \quad , \quad \frac{d\alpha}{dy} < 0$$

$$y > c \quad , \quad \frac{d\alpha}{dy} > 0$$

As shown in Figure D-1d, three regions are obtained with discontinuities between them.

The third case has $c > 0$ and $\lambda c+r = 0$. Here

$c = \frac{-r}{\lambda}$, so the only possibility is $\lambda < 0$. The slope is infinite at $y = 0$ and at first sight is indeterminate at $y = c$. Application of L'Hospital's Rule, however, shows that $\frac{d\alpha}{dy} = \frac{1}{r}$ at $y = c$. In this case, no maxima or minima exist except at infinity. The slope is positive for $y > 0$ and negative for $y < 0$, as indicated in Figure D-1e.

Whenever $c > 0$ and $\lambda c + r \gtrless 0$, it is not possible to have a continuous solution in the range $0 \leq \alpha \leq 1$. Since $c \equiv \frac{1}{2}\left[y(o) + y(1)\right]$ and $y(1) < y(o)$ for a compressive shock, it is necessary that $y(o) > c$ and $y(1) < c$. It is not possible to have a continuous solution without going outside of the range $0 \leq \alpha \leq 1$.

If $c > 0$ and $\lambda c + r = 0$, a continuous solution can be found between $y(1)$ and $y(o)$. The slope is positive, however, so that $y(1) > y(o)$ which would indicate an expansion shock.

The possibility $c > 0$ is ruled out, therefore, by not allowing discontinuities within the range $0 \leq \alpha \leq 1$ and by not allowing expansion shocks.

The sketches of the solutions for all the subcases within the case $c < 0$ are obtained in similar fashion and are shown in Figure D-2. Identical arguments may obviously be used to rule out the possibility $c < 0$.

Finally, it can be shown that in the case $c = 0$, the restrictions on the solution are not violated. In that case, the Equation (D-1) becomes

$$\frac{d\alpha}{dy} = \frac{-1}{\lambda y + r} \qquad \text{(D-2)}$$

There are three subcases: $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$. If $\lambda \neq 0$, an infinite slope occurs at $y = -\frac{r}{\lambda}$. Whenever $\lambda = 0$, a straight line is obtained. The curves for the three subcases are shown in Figure D-3. Note that in all subcases, a continuous solution may be found with $y(o) > c = 0 > y(1)$. It is concluded, therefore, that $c = 0$ is the only possibility.

Now, it remains to solve Equation (D-2) with the condition $y(1) = -y(o)$. This statement of a relationship between the solutions at two points is just as satisfactory in determining the arbitrary constant as is the statement of the exact value of the solution at one point. Equation (D-2) may be integrated to yield the following result:

$$\alpha = -\int_{y(o)}^{y} \frac{dy}{\lambda y + r} = -\frac{1}{\lambda} \ln\left(\frac{\lambda y + r}{\lambda y(o) + r}\right)$$

This may be rewritten in a more convenient form as follows:

$$y(\alpha) = -\frac{r}{\lambda} + \left[\frac{r}{\lambda} + y(o)\right] e^{-\lambda\alpha} \qquad \text{(D-3)}$$

Setting $\alpha = 1$, it is found that

$$y(o) = -y(1) = \frac{r}{\lambda}\,\frac{1-e^{-\lambda}}{1+e^{-\lambda}}$$

Substituting the above for $y(0)$ in Equation (D-3), the final solution is:

$$Q_1(\alpha) \equiv y(\alpha) = \frac{r}{\lambda}\left[\frac{2}{1+e^{-\lambda}}\, e^{-\lambda\alpha} - 1\right]$$

Taking the limit as $\lambda \longrightarrow 0$, a linear relation is found:

$$Q_1(\alpha) = r\left(\frac{1}{2} - \alpha\right)$$

This same result could have been obtained by setting $\lambda = 0$ in the differential Equation (D-2) and then integrating.

APPENDIX E

NONLINEAR EXTENSION OF CROCCO TIME-LAG POSTULATE

According to Crocco (Ref. 1) the combustion process for a particular element of propellant is sensitive to perturbations in pressure and temperature over a time period of magnitude τ before that instant when the propellant initially becomes burned gas. The combustion process occurs at the rate f during the period τ until an amount E is accumulated. That is, for an element of propellant burning at time t,

$$E = \int_{t-\tau}^{t} f \, dt' \qquad \text{(E-1)}$$

In other words, f is the rate of accumulation of some entity and E is the amount of that entity which must be accumulated before the element burns. f will be sensitive to both pressure and temperature fluctuations.

The exact nature of this "entity" is unknown and also the specific relationship between f and temperature and pressure is not certain. This will be left in an arbitrary form. Much may still be determined from this type of approach as has been accomplished for the linear case by Crocco.

It is convenient to relate the mass flux of burned gases to the rate of change of time-lag of the propellant burning at a particular instant. Note that the assumption of concentrated combustion zone at the injector end is used. The concept, however, is readily extendable to other cases (see Ref. 1). The mass flux $\dot{m}_1$ of unburned propellants entering the pressure sensitive portion of the combustion process at time $t-\tau$ is different from the mass flux of burned gases, $\dot{m}_b$ emitted from the combustion zone at time t. (The mass flux entering this sensitive portion of the combustion process equals the mass flux injected even in the unsteady-state). If δt_2 is the time during which a unit mass of unburned propellants enters the pressure-sensitive portion of the combustion process and if δt_1 is the time during which the same unit mass is emitted as burned gas, the conservation of mass relation tells us that

$$\dot{m}_b(t)\,\delta t_1 = \dot{m}_i(t-\tau)\,\delta t_2 \tag{E-2}$$

The difference between the two time durations (δt_1 and δt_2) depends upon the difference in the period τ of the particle at the beginning of the duration and the period τ of the particle at the end of the duration. The relation is

$$\delta t_1 = \delta t_2 + \tau(t+\delta t_1) - \tau(t) \tag{E-3}$$

where $\tau(t)$ is the duration of the pressure-sensitive portion of the combustion process for a particle which burns at time t. Equations (E-2) and (E-3) may be combined to yield the relation

$$\frac{\dot{m}_b}{\dot{m}_i} = \frac{\delta t_2}{\delta t_1} = 1 - \frac{\tau(t+\delta t_1) - \tau(t)}{\delta t_1}$$

Taking the limit as $\delta t_1 \longrightarrow 0$, we get

$$\frac{\dot{m}_b}{\dot{m}_i} = 1 - \frac{d\tau}{dt} \tag{E-4}$$

Now, it is desirable to relate $\frac{d\tau}{dt}$ to fluctuations in the thermodynamics properties by means of the relation given by Equation (E-1) and, also, a relation between f and the thermodynamic properties. The resulting relationship would provide the boundary condition at one chamber-end for the equations governing the wave phenomenon. Since the wave equation is being solved by a perturbation technique, it is convenient to express the boundary condition as a series* in the amplitude parameter, ϵ.

Also, the boundary condition should be transformed to apply in the characteristic coordinates which are used in the gasdynamic analysis of Chapter III.

It is seen that the duration of the period of pressure and temperature sensitivity, τ, and the rate function, f, are both dependent upon the amplitude of the fluctuation. The critical amount of the entity-to-be-accumulated, E, does not depend on the amplitude; i.e., it is a constant associated with the particular injector and propellants. Expanding the right-hand side of Equation (E-1) in a Taylor series, a relation is obtained as follows:

$$E = \int_{t-\tau(t,\epsilon)}^{t} f(t',\epsilon)\,dt' = \int_{t-\tau(t,0)}^{t} f(t',0)\,dt' + \epsilon\left[\frac{\partial}{\partial\epsilon}\int_{t-\tau(t,\epsilon)}^{t} f(t',\epsilon)\,dt'\right]_{\epsilon=0} +$$

$$+\frac{\epsilon^2}{2!}\left[\frac{\partial^2}{\partial\epsilon^2}\int_{t-\tau(t,\epsilon)}^{t} f(t',\epsilon)\,dt'\right]_{\epsilon=0} + \frac{\epsilon^3}{3!}\left[\frac{\partial^3}{\partial\epsilon^3}\int_{t-\tau(t,\epsilon)}^{t} f(t',\epsilon)\,dt'\right]_{\epsilon=0}$$

$$+\,O(\epsilon^4)$$

Since E is independent of ϵ, it is concluded that $E = \int_{t-\tau(t,0)}^{t} f(t',0)\,dt'$.

Combining this statement with the above, the simplified relation

* The series must include terms of order ϵ^3 and below since, as will later be shown, the eigenvalue perturbations are of second order.

is written:

$$0=\epsilon\left[\frac{\partial}{\partial\epsilon}\int_{t-\tau(t,\epsilon)}^{t} f(t',\epsilon)\,dt'\right]_{\epsilon=0} + \frac{\epsilon^2}{2!}\left[\frac{\partial^2}{\partial\epsilon^2}\int_{t-\tau(t,\epsilon)}^{t} f(t',\epsilon)\,dt'\right]_{\epsilon=0} +$$

$$+\frac{\epsilon^3}{3!}\left[\frac{\partial^3}{\partial\epsilon^3}\int_{t-\tau(t,\epsilon)}^{t} f(t',\epsilon)\,dt'\right]_{\epsilon=0} + O(\epsilon^4) \quad \text{(E-5)}$$

Equation (E-5) states that f and τ perturbations must be related to each other in such a manner that the integral of Equation (E-1) remains unchanged. For example, if the average value of f over the time period $t-\tau \leq t' \leq t$ is less than the steady-state value, τ is then greater than its steady-state value.

Leibnitz' Rule is used to evaluate the derivatives of the integrals in Equation (E-5). Suppose that $f(t,\epsilon)$ and may be written in series form*:

$$f = f_0 + \epsilon f_1(t) + \epsilon^2 f_2(t) + \epsilon^3 f_3(t) + O(\epsilon^4)$$

$$\tau = \tau_0 + \epsilon\tau_1(t) + \epsilon^2\tau_2(t) + \epsilon^3\tau_3(t) + O(\epsilon^4) \quad \text{(E-6)}$$

If we assume that the series are convergent, it follows that

$$f_0 = (f)_{\epsilon=0};\; f_1 = \left(\frac{\partial f}{\partial\epsilon}\right)_{\epsilon=0};\; f_2 = \frac{1}{2!}\left(\frac{\partial^2 f}{\partial\epsilon^2}\right)_{\epsilon=0};\; f_3 = \frac{1}{3!}\left(\frac{\partial^3 f}{\partial\epsilon^3}\right)_{\epsilon=0}$$

$$\tau_0 = (\tau)_{\epsilon=0};\; \tau_1 = \left(\frac{\partial\tau}{\partial\epsilon}\right)_{\epsilon=0};\; \tau_2 = \frac{1}{2!}\left(\frac{\partial^2\tau}{\partial\epsilon^2}\right)_{\epsilon=0};\; \tau_3 = \frac{1}{3!}\left(\frac{\partial^3\tau}{\partial\epsilon^3}\right)_{\epsilon=0}$$

Using this information, the application of Leibnitz' Rule to

* Note that it is always assumed that $f_0(t')$ may be replaced by its average value over the period $t-\tau \leq t' \leq t$. So, f_0 is a constant independent of time.

Equation (E-5) yields the following:

$$0=\epsilon\left[\int_{t-\tau_0}^{t} f_1\,dt'+f_0\,\tau_1\right]+$$
$$+\epsilon^2\left[\int_{t-\tau_0}^{t} f_2\,dt'+f_0\,\tau_2+f_1(t-\tau_0)\cdot\tau_1\right]+$$
$$+\epsilon^3\left[\int_{t-\tau_0}^{t} f_3\,dt'+f_2(t-\tau_0)\cdot\tau_1+f_1(t-\tau_0)\cdot\tau_2+f_0\cdot\tau_3\right.$$
$$\left.-\tfrac{1}{2}(\tau_1)^2\,\frac{df_1}{dt}(t-\tau_0)\right]+o(\epsilon^4)$$

An expression for $f_0[\epsilon\,\tau_1+\epsilon^2\tau_2+\epsilon^3\tau_3]$ is readily obtainable from the above equation. However, this will not be an explicit relationship between the perturbation in τ and the perturbation in f since such terms as $f_1(t-\tau_0)\cdot[\epsilon^2\tau_1+\epsilon^3\tau_2+o(\epsilon^4)]$ will appear. This appearance of the coefficients τ_1, τ_2 etc., are in terms of higher order in ϵ, so that the method of successive substitutions is applied to obtain an explicit relationship*. This relation may be written as follows:

$$-f_0[\epsilon\,\tau_1+\epsilon^2\tau_2+\epsilon^3\tau_3]=\epsilon\left[\int_{t-\tau_0}^{t} f_1\,dt'\right]+$$
$$+\epsilon^2\left[\int_{t-\tau_0}^{t} f_2\,dt'-f_1(t-\tau_0)\frac{1}{f_0}\int_{t-\tau_0}^{t} f_1\,dt'\right]+$$
$$+\epsilon^3\left[\int_{t-\tau_0}^{t} f_3\,dt'-f_2(t-\tau_0)\frac{1}{f_0}\int_{t-\tau_0}^{t} f_1\,dt'-\right.$$
$$-f_1(t-\tau_0)\frac{1}{f_0}\int_{t-\tau_0}^{t} f_2\,dt'+\left(\frac{f_1(t-\tau_0)}{f_0}\right)^2\int_{t-\tau_0}^{t} f_1\,dt'$$
$$\left.-\tfrac{1}{2}\left(\frac{1}{f_0}\int_{t-\tau_0}^{t} f_1\,dt'\right)^2\frac{df_1}{dt}(t-\tau_0)\right]+o(\epsilon^4)$$

*Separation of the equations according to powers in ϵ is withheld temporarily until coordinate perturbations and eigenvalue perturbations are completed.

The relation on the previous page may be differentiated to obtain $\frac{d\tau}{dt}$ (Note that $\frac{d\tau_0}{dt} = 0$).

$$\frac{d\tau}{dt} = \epsilon \left[\frac{f_1(t-\tau_0) - f_1(t)}{f_0} \right] + $$

$$+ \epsilon^2 \left[\frac{f_2(t-\tau_0) - f_2(t)}{f_0} + \frac{f_1(t) f_1(t-\tau_0) - f_1^2(t-\tau_0)}{f_0^2} + \right.$$

$$\left. + \frac{1}{f_0^2} \frac{df_1}{dt}(t-\tau_0) \int_{t-\tau_0}^{t} f_1 \, dt' \right] + \epsilon^3 \left[\frac{f_3(t-\tau_0) - f_3(t)}{f_0} + \right.$$

$$+ \frac{1}{f_0^2} \frac{df_2}{dt}(t-\tau_0) \int_{t-\tau_0}^{t} f_1 \, dt' + \frac{1}{f_0^2} f_2(t-\tau_0) \left\{ f_1(t) - f_1(t-\tau_0) \right\} +$$

$$+ \frac{1}{f_0^2} \frac{df_1}{dt}(t-\tau_0) \left\{ \int_{t-\tau_0}^{t} f_2 \, dt' - 2 \frac{f_1(t-\tau_0)}{f_0} \int_{t-\tau_0}^{t} f_1 \, dt' + \right.$$

$$\left. + \frac{f_1(t) - f_1(t-\tau_0)}{f_0} \int_{t-\tau_0}^{t} f_1 \, dt' \right\} + \frac{f_1(t-\tau_0)}{f_0} \left\{ \frac{f_2(t) - f_2(t-\tau_0)}{f_0} - \right.$$

$$\left. \left. - \frac{f_1(t) f_1(t-\tau_0) - f_1^2(t-\tau_0)}{f_0^2} \right\} + \frac{1}{2 f_0^3} \left(\int_{t-\tau_0}^{t} f_1 \, dt' \right)^2 \frac{d^2 f_1}{dt^2}(t-\tau_0) \right] + O(\epsilon^4) \tag{E-7}$$

Now, the rate function f depends upon temperature and pressure. It will be assumed that the dependence is the same throughout the combustion process; i.e., $f(p(t'), T(t'))$ would remain constant with p and T even if t' varied. In other words, the effect (upon f) of changes in thermodynamic conditions is equally important for each and every instant a particle spends in the process of being combusted. Furthermore, by means of an equation of state and the Laws of Thermodynamics, the pressure and temperature at any time may be related to the speed of sound* at that time. The exact nature of this relationship can only be determined by an extensive investigation of the combustion

* It is convenient to work with the speed of sound when analyzing wave phenomena.

process. For our purposes, an arbitrary relationship between f and the speed of sound a shall be assumed and the theory shall be developed in this heuristic manner. Specifically, it is said that a Taylor series may be written

$$\frac{f(a)}{f(1)} = 1 + N(a-1) + M(a-1)^2 + $$
$$+ L(a-1)^3 + \text{Order } (a-1)^4$$

where

$$N \equiv \left(\frac{1}{f}\frac{df}{da}\right)_{a=1} \; ; \; M \equiv \left(\frac{1}{2f}\frac{d^2f}{da^2}\right)_{a=1} \; ; \; L \equiv \left(\frac{1}{6f}\frac{d^3f}{da^3}\right)_{a=1}$$

The standard nondimensional scheme is used (see Chapter I, page 13) so that a = 1 is the steady-state value for the speed of sound. N, M, and L are coefficients which are related to the combustion process and they must remain arbitrary until the exact nature of the combustion process is understood. However, the important point is that stability criteria are strongly related to these coefficients as is shown in Chapter III. The definition $n = \frac{\gamma-1}{2\gamma}N$ is made to be consistent with Crocco (Ref. 1) who wrote f as a Taylor series in the pressure perturbation $\left(n \equiv \left(\frac{1}{f}\frac{df}{dp}\right)_{p=1}\right)$. As stated in Chapter III (Equation 2), the speed of sound is written as a series in ϵ. Substitution of this perturbation series in the above Taylor series for f (a) and comparison with Equation (E-6) yields the following information:

$$\frac{f_1}{f_0} = \frac{2\gamma n}{\gamma-1} a_1 \; ; \; \frac{f_2}{f_0} = \frac{2\gamma n}{\gamma-1} a_2 + M a_1^2$$

$$\frac{f_3}{f_0} = \frac{2\gamma n}{\gamma-1} a_3 + 2 M a_1 a_2 + L a_1^3$$

The relations on the previous page are substituted into Equation (E-7) with the result

$$
\begin{aligned}
\frac{d\tau}{dt} = {} & \epsilon\left\{\frac{2\gamma n}{\gamma-1}\left[a_1(t-\tau_0)-a_1(t)\right]\right\}+ \\
& +\epsilon^2\left\{\frac{2\gamma n}{\gamma-1}\left[a_2(t-\tau_0)-a_2(t)\right]+M\left[a_1^2(t-\tau_0)-a_1^2(t)\right]\right. \\
& \left.+\left(\frac{2\gamma n}{\gamma-1}\right)^2 a_1(t-\tau_0)\left[a_1(t)-a_1(t-\tau_0)\right]+\left(\frac{2\gamma n}{\gamma-1}\right)^2\frac{da_1}{dt}(t-\tau_0)\int_{t-\tau_0}^{t}a_1\,dt'\right\} \\
& +\epsilon^3\left\{\frac{2\gamma n}{\gamma-1}\left[a_3(t-\tau_0)-a_3(t)\right]+2\left[M-\left(\frac{2\gamma n}{\gamma-1}\right)^2\right]a_2(t-\tau_0)\,a_1(t-\tau_0)\right. \\
& +\left(\frac{2\gamma n}{\gamma-1}\right)^2 a_2(t-\tau_0)\,a_1(t)+\left(\frac{2\gamma n}{\gamma-1}\right)^2 a_2(t)\,a_1(t-\tau_0)-2Ma_2(t)\,a_1(t) \\
& +\left[L-\frac{4\gamma n}{\gamma-1}M+\left(\frac{2\gamma n}{\gamma-1}\right)^3\right]a_1^3(t-\tau_0)+\left[\frac{2\gamma n}{\gamma-1}M-\left(\frac{2\gamma n}{\gamma-1}\right)^3\right]a_1^2(t-\tau_0)\,a_1(t) \\
& +\frac{2\gamma n}{\gamma-1}M\,a_1^2(t)\,a_1(t-\tau_0)-La_1^3(t)+\left(\frac{2\gamma n}{\gamma-1}\right)^2\frac{da_2}{dt}(t-\tau_0)\int_{t-\tau_0}^{t}a_1\,dt' \\
& +\left[\frac{4\gamma n}{\gamma-1}M-3\left(\frac{2\gamma n}{\gamma-1}\right)^3\right]a_1(t-\tau_0)\frac{da_1}{dt}(t-\tau_0)\int_{t-\tau_0}^{t}a_1\,dt'+\left(\frac{2\gamma n}{\gamma-1}\right)^2\frac{da_1}{dt}(t-\tau_0)\int_{t-\tau_0}^{t}a_2\,dt'+ \\
& +\frac{2\gamma n}{\gamma-1}M\frac{da_1}{dt}(t-\tau_0)\int_{t-\tau_0}^{t}a_1^2\,dt'+\frac{1}{2}\left(\frac{2\gamma n}{\gamma-1}\right)^3\frac{d^2}{dt^2}a_1(t-\tau_0)\left[\int_{t-\tau_0}^{t}a_1\,dt'\right]^2 \\
& \left.+\left(\frac{2\gamma n}{\gamma-1}\right)^3\frac{da_1}{dt}(t-\tau_0)\,a_1(t)\int_{t-\tau_0}^{t}a_1\,dt'\right\}
\end{aligned}
\qquad \text{(E-8)}
$$

The mass flux from the combustion zone should also be expanded in a series in ϵ before substituting Equation (E-8) into Equation (E-4). Of course, the mass flux $\dot{m} = \rho u$. Furthermore, under steady-state operation, the flux injected equals the flux emitted from the combustion zone. So, $\dot{m}_i = \rho_o u_o$. Noting that both ρ and u may be expressed as series in ϵ, we find the following

$$\frac{\dot{m}_b}{\dot{m}_i} = \frac{\rho u}{\rho_o u_o} = 1 + \epsilon\left(\rho_1 + \frac{u_1}{u_o}\right) + \epsilon^2\left(\rho_2 + \rho_1\frac{u_1}{u_o} + \frac{u_2}{u_o}\right) + \epsilon^3\left(\rho_3 + \rho_2\frac{u_1}{u_o} + \rho_1\frac{u_2}{u_o} + \frac{u_3}{u_o}\right) + O(\epsilon^4) \qquad \text{(E-9)}$$

It is convenient to relate the coefficients ρ_1, ρ_2, etc. to the coefficients a_1, a_2, etc. Under the assumptions that the gas is perfect and the flow field is homentropic* outside of the combustion zone, the following statement is valid:

$$\rho = a^{\frac{2}{\gamma-1}}$$

Substitution of the perturbation series for both ρ and a, expansion of the right-hand side in a binomial series, and separation according to powers in ϵ yield the results:

$$\rho_1 = \frac{2}{\gamma-1}\,a_1 \quad ; \quad \rho_2 = \frac{2}{\gamma-1}\,a_2 + \frac{3-\gamma}{(\gamma-1)^2}\,a_1^2$$

$$\rho_3 = \frac{2}{\gamma-1}\,a_3 + 2\,\frac{(3-\gamma)}{(\gamma-1)^2}\,a_1 a_2 + \frac{2}{3}\,\frac{(3-\gamma)(2-\gamma)}{(\gamma-1)^3}\,a_1^3$$

Combination of the above information and Equations (E-4),

*In reality an entropy wave would be created, but its effect is neglected.

(E-8), and (E-9) produces a relationship between the perturbation of gas velocity u and the perturbation of the speed of sound a at the end of the combustion zone. This boundary condition is stated as

$$\sum_{i=1}^{\infty} \epsilon^i \left[\frac{2}{\gamma-1}(1-\gamma n)a_i(t)+\frac{u_i(t)}{u_0}+\frac{2\gamma n}{\gamma-1}\, a_i\,(t-\tau_0)\right] = \sum_{i=1}^{\infty} \epsilon^i R_i''$$

where the inhomogeneous terms are defined as follows:

$$R_1'' \equiv 0$$

$$\begin{aligned}
R_2'' \equiv {} & \left[M - \frac{3-\gamma}{(\gamma-1)^2} + \left(\frac{2}{\gamma-1}\right)^2 (1-\gamma n)\right] a_1^2(t) + \\
& + \left[\left(\frac{2\gamma n}{\gamma-1}\right)^2 - M\right] a_1^2(t-\tau_0) + \left[\frac{4\gamma n}{(\gamma-1)^2} - \left(\frac{2\gamma n}{\gamma-1}\right)^2\right] a_1(t)\, a_1(t-\tau_0) \\
& - \left(\frac{2\gamma n}{\gamma-1}\right)^2 \frac{da_1}{dt}(t-\tau_0) \int_{t-\tau_0}^{t} a_1\, dt'
\end{aligned}$$

$$\begin{aligned}
R_3'' \equiv {} & 2\left[\left(\frac{2\gamma n}{\gamma-1}\right)^2 - M\right] a_2(t-\tau_0)\, a_1(t-\tau_0) - \left(\frac{2\gamma n}{\gamma-1}\right)^2 a_2(t-\tau_0)\, a_1(t) \\
& - \left(\frac{2\gamma n}{\gamma-1}\right)^2 a_2(t)\, a_1(t-\tau_0) + 2\left[M - \frac{3-\gamma}{(\gamma-1)^2}\right] a_2(t)\, a_1(t) \\
& + \left[\frac{4\gamma n}{\gamma-1} M - \left(\frac{2\gamma n}{\gamma-1}\right)^3 - L\right] a_1^3(t-\tau_0) + \left[\left(\frac{2\gamma n}{\gamma-1}\right)^3 - \frac{2\gamma n}{\gamma-1} M\right] a_1^2(t-\tau_0)\, a_1(t) \\
& - \frac{2\gamma n M}{\gamma-1}\, a_1^2(t)\, a_1(t-\tau_0) + \left[L - \frac{2}{3}\, \frac{(3-\gamma)(2-\gamma)}{(\gamma-1)^3}\right] a_1^3(t) \\
& - \frac{2}{\gamma-1}\, a_2(t)\, \frac{u_1}{u_0}(t) - \frac{3-\gamma}{(\gamma-1)^2}\, a_1^2(t)\, \frac{u_1(t)}{u_0} - \frac{2}{\gamma-1}\, a_1(t)\, \frac{u_2(t)}{u_0} - \\
& - \left(\frac{2\gamma n}{\gamma-1}\right)^2 \frac{da_2}{dt}(t-\tau_0) \int_{t-\tau_0}^{t} a_1\, dt' + \left[3\left(\frac{2\gamma n}{\gamma-1}\right)^3 - \frac{4\gamma n M}{\gamma-1}\right] a_1(t-\tau_0)\, \frac{da_1}{dt}(t-\tau_0) \int_{t-\tau_0}^{t} a_1\, dt' \\
& - \left(\frac{2\gamma n}{\gamma-1}\right)^2 \frac{da_1}{dt}(t-\tau_0) \int_{t-\tau_0}^{t} a_2\, dt' - \frac{2\gamma n}{\gamma-1} M\, \frac{da_1}{dt}(t-\tau_0) \int_{t-\tau_0}^{t} a_1^2\, dt' - \\
& - \frac{1}{2}\left(\frac{2\gamma n}{\gamma-1}\right)^3 \frac{d^2 a_1}{dt^2}(t-\tau_0) \left[\int_{t-\tau_0}^{t} a_1\, dt'\right]^2 - \left(\frac{2\gamma n}{\gamma-1}\right)^3 a_1(t)\, \frac{da_1}{dt}(t-\tau_0) \int_{t-\tau_0}^{t} a_1\, dt'
\end{aligned}$$

(E-10)

If $\epsilon \to 0$, this boundary condition agrees with Crocco's result for linear perturbations which states

$$\frac{2}{\gamma-1}(1-\gamma n)\, a'(t) + \frac{u'}{u_0}(t) + \frac{2\gamma n}{\gamma-1}\, a'(t-\tau_0) = 0$$

where primes denote infinitesimal perturbations. The appearance of the retarded variable $a(t-\tau_0)$ under the perturbation scheme is the reason for the name "time-lag theory."

The boundary condition represented by Equations (E-10) is given in a time coordinate system. The wave problem is solved in characteristic coordinates, however, so it is necessary to rewrite the boundary condition in the new coordinate system. Care must be taken in transforming the retarded variable $a(t-\tau_0)$ to characteristic coordinates. While the time-lag τ_0 is a constant in time coordinates, it is not a constant in characteristic coordinates. Due to wave distortion, the time-lag is an oscillating quantity. Wave distortion is a nonlinear effect, however, so that the oscillation of the time-lag is of higher order in ϵ and may be taken into account in an orderly fashion.

As shown in Chapter III, space and time coordinates are rewritten as functions of the new independent variables, the characteristic coordinates α and β, and, furthermore, are perturbed as a series in ϵ. At the combustion zone, in particular, $\alpha = \beta$ and we have

$$\begin{aligned} x_0(\beta,\beta) &= x_1(\beta,\beta) = x_2(\beta,\beta) = 0 \\ t(\beta,\beta) &= t_0(\beta,\beta) + \epsilon t_1(\beta,\beta) + \epsilon^2 t_2(\beta,\beta) + O(\epsilon^3) \end{aligned} \quad \text{(E-11)}$$

Consider $t^* = t - \tau_o$ and let β^* correspond to t^* such that

$$t^*(\beta^*, \beta^*) = t(\beta, \beta) - \tau_o \tag{E-12}$$

It is desired to relate β^* to β. The difference is the lag in characteristic coordinates. Substituting the series from Equation (E-11) into Equation (E-12), we have

$$\tau_o = \left[t_o(\beta,\beta) - t_o(\beta^*,\beta^*)\right] + \epsilon\left[t_1(\beta,\beta) - t_1(\beta^*,\beta^*)\right] + \\ + \epsilon^2\left[t_2(\beta,\beta) - t_2(\beta^*,\beta^*)\right] + O(\epsilon^3) \tag{E-13}$$

The zero order solution for $t_o(\alpha,\beta)$ is obtained in Chapter III and is written as follows

$$t_o(\beta,\beta) = \frac{2\beta}{1-u_o^2}$$

$$t_o(\beta^*,\beta^*) = \frac{2\beta^*}{1-u_o^2}$$

A convenient definition is made

$$b_o \equiv \frac{1-u_o^2}{2}\tau_o$$

These statements are combined with Equation (E-13) to yield

$$\beta^* = \beta - b_o + \epsilon\frac{1-u_o^2}{2}\left[t_1(\beta,\beta) - t_1(\beta^*,\beta^*)\right] + \\ + \epsilon^2\frac{1-u_o^2}{2}\left[t_2(\beta,\beta) - t_2(\beta^*,\beta^*)\right] + O(\epsilon^3) \tag{E-14}$$

Using a Taylor series expansion for $t_1(\beta^*,\beta^*)$ about the point

$\beta - b_o, \beta - b_o$, we see that

$$t_1(\beta^*, \beta^*) = t_1(\beta - b_o, \beta - b_o) + \left(\frac{dt_1}{d\beta}\right)_{\beta - b_o}\left[\beta^* - \beta + b_o\right] + \\ + O(\beta^* - \beta + b_o)^2$$

Similarly, we have $t_2(\beta^*, \beta^*) = t_2(\beta - b_o, \beta - b_o) + O(\beta^* - \beta + b_o)$

Note that the derivative is taken along the line $\alpha = \beta$.

Combining the two above relations with Equation (E-14) by successive substitutions, we find

$$t_1(\beta^*, \beta^*) = t_1(\beta - b_o, \beta - b_o) + \epsilon \frac{1-u_o^2}{2}\frac{dt_1}{d\beta}(\beta - b_o, \beta - b_o)\left[t_1(\beta, \beta) - \\ - t_1(\beta - b_o, \beta - b_o)\right] + O(\epsilon^2)$$

$$t_2(\beta^*, \beta^*) = t_2(\beta - b_o, \beta - b_o) + O(\epsilon)$$

Now, Equation (E-14) becomes

$$\beta^* = \beta - b_o + \epsilon\frac{1-u_o^2}{2}\left[t_1(\beta,\beta) - t_1(\beta - b_o, \beta - b_o)\right] + \epsilon^2\frac{1-u_o^2}{2}\left[t_2(\beta,\beta) - t_2(\beta - b_o, \beta - b_o)\right] \\ - \epsilon^2\left(\frac{1-u_o^2}{2}\right)^2\frac{dt_1}{d\beta}(\beta - b_o, \beta - b_o)\left[t_1(\beta,\beta) - t_1(\beta - b_o, \beta - b_o)\right] + O(\epsilon^3)$$

(E-15)

Consider an arbitrary retarded variable $\xi(\beta^*, \beta^*)$ (e.g., $a_1(\beta^*, \beta^*)$). Using a Taylor series expansion about the point β-b_o, β-b_o along with Equation (E-15) for the difference between β^* and β-b_o, we have

$$\xi(\beta^*, \beta^*) = \xi(\beta - b_o, \beta - b_o) + \epsilon\frac{1-u_o^2}{2}\frac{d\xi}{d\beta}(\beta - b_o, \beta - b_o)\left[t_1(\beta,\beta) - \\ - t_1(\beta - b_o, \beta - b_o)\right] + \epsilon^2\frac{1-u_o^2}{2}\frac{d\xi}{d\beta}(\beta - b_o, \beta - b_o)\left[t_2(\beta,\beta) - t_2(\beta - b_o, \beta - b_o)\right] \\ - \epsilon^2\frac{d\xi}{d\beta}(\beta - b_o, \beta - b_o)\left(\frac{1-u_o^2}{2}\right)^2\frac{dt_1}{d\beta}(\beta - b_o, \beta - b_o)\left[t_1(\beta,\beta) - t_1(\beta - b_o, \beta - b_o)\right] \\ + \frac{\epsilon^2}{2}\frac{d^2\xi}{d\beta^2}(\beta - b_o, \beta - b_o)\left(\frac{1-u_o^2}{2}\right)^2\left[t_1(\beta,\beta) - t_1(\beta - b_o, \beta - b_o)\right]^2 + O(\epsilon^3)$$

(E-16)

Now, this relation may be used in a straightforward manner to evaluate the $a_i(t-\tau_0) = a_i(\beta^*, \beta^*)$ in Equations (E-10). Note that any changes of order ϵ^4 produced by the coordinate transformation will be neglected. In view of these considerations, there still remains one term to be transformed. This is the last term in the expression for R_2''. It essentially involves the term

$$\frac{da_1}{dt}(t-\tau_0)\int_{t-\tau_0}^{t} a_1\, dt'.$$

This may be rewritten as

$$\frac{\frac{da_1}{d\beta}(\beta^*, \beta^*)}{\frac{dt}{d\beta}(\beta^*, \beta^*)}\int_{\beta^*}^{\beta} a_1 \frac{dt}{d\beta}\, d\beta'$$

The following is readily seen from Equations (2) and (10) in Chapter III

$$\frac{dt}{d\beta}(\beta,\beta) = \frac{dt_0}{d\beta}(\beta,\beta) + \epsilon\frac{dt}{d\beta}(\beta,\beta) + O(\epsilon^2)$$

$$= \frac{2}{1-u_0^2} + \epsilon\frac{dt_1}{d\beta}(\beta,\beta) + O(\epsilon^2)$$

Furthermore, we see that

$$\frac{1}{\frac{dt}{d\beta}(\beta^*, \beta^*)} = \frac{1}{\frac{dt_0}{d\beta}(\beta_1^*, \beta^*)}\left[1 - \epsilon\frac{\frac{dt_1}{d\beta}(\beta^*, \beta^*)}{\frac{dt_0}{d\beta}(\beta_1^*\beta^*)} + O(\epsilon^2)\right]$$

$$= \frac{1-u_0^2}{2}\left[1 - \epsilon\frac{1-u_0^2}{2}\frac{dt_1}{d\beta}(\beta^*, \beta^*) + O(\epsilon^2)\right]$$

Now, treating the integral, it is found that

$$\int_{\beta^*}^{\beta} a_1\frac{dt}{d\beta}\, d\beta' = \frac{2}{1-u_0^2}\left[\int_{\beta-b_0}^{\beta} a_i\, d\beta' + \int_{\beta^*}^{\beta-b_0} a_1\, d\beta'\right] + \epsilon\int_{\beta^*}^{\beta} a_1\frac{dt_1}{d\beta}\, d\beta' +$$

$$+ O(\epsilon^2)$$

Note that $\int_{\beta^*}^{\beta-b_0} a_1\, d\beta' = a_1(\beta-b_0, \beta-b_0)[\beta-b_0-\beta^*] + 0[\beta-b_0-\beta^*]^2$

The same result would be obtained by expansion in a Taylor series with the lower integral limit as the expansion parameter. Combining this with Equation (E-15), the integral is evaluated as follows

$$\int_{\beta^*}^{\beta} a_1 \frac{dt}{d\beta}\, d\beta' = \frac{2}{1-u_0^2}\int_{\beta-b_0}^{\beta} a_1\, d\beta' - \epsilon\, a_1(\beta-b_0, \beta-b_0)[t_1(\beta,\beta) - t_1(\beta-b_0, \beta-b_0)]$$

$$+\epsilon \int_{\beta-b_0}^{\beta} a_1 \frac{dt_1}{d\beta}\, d\beta' + 0(\epsilon^2)$$

Considering $\frac{da_1}{d\beta}(\beta^*, \beta^*) = \xi(\beta^*, \beta^*)$, we may use Equation (E-16) to obtain the proper transformation for this derivative. Now, the term with the time derivative and the time integral may be readily evaluated in a perturbation series form.

The coordinate transformation can now be applied to Equation (E-10). The short-hand notations $a_i \equiv a_i(\beta, \beta)$ and $a_{i_{b_0}} \equiv (\beta-b_0, \beta-b_0)$ are used in this final expression

$$\sum_{i=1}^{\infty} \epsilon^i \left[\frac{2}{\gamma-1}(1-\gamma n) a_i + \frac{u_i}{u_0} + \frac{2\gamma n}{\gamma-1} a_{i_{b_0}}\right] = \sum_{i=1}^{\infty} \epsilon^i R_i'$$

where the inhomogeneous terms* are

$$R_2' = \left[M - \frac{3-\gamma}{(\gamma-1)^2}\right] a_1^2 - \frac{2}{\gamma-1} a_1 \frac{u_1}{u_0} +$$

$$+ \left[\left(\frac{2\gamma n}{\gamma-1}\right)^2 - M\right] a_{1_{b_0}}^2 - \left(\frac{2\gamma n}{\gamma-1}\right)^2 a_{1_{b_0}} a_1$$

$$- \left(\frac{2\gamma n}{\gamma-1}\right)^2 \frac{da_1}{d\beta}_{b_0} \int_{\beta-b_0}^{\beta} a_1\, d\beta' - \frac{\gamma n}{\gamma-1} \frac{da_1}{d\beta}_{b_0} (1-u_0^2)[t_1 - t_{1_{b_0}}]$$

*Note that $R_1' = 0$

$$
\begin{aligned}
R_3' \equiv\ & 2\left[\left(\frac{2\gamma n}{\gamma-1}\right)^2 - M\right] a_{2_{b_0}}\, a_{1_{b_0}} - \left(\frac{2\gamma n}{\gamma-1}\right)^2 a_{2_{b_0}}\, a_1 \\
& - \left(\frac{2\gamma n}{\gamma-1}\right)^2 a_2\, a_{1_{b_0}} + 2\left[M - \frac{3-\gamma}{(\gamma-1)^2}\right] a_2\, a_1 + \\
& + \left[\frac{4\gamma n M}{\gamma-1} - \left(\frac{2\gamma n}{\gamma-1}\right)^3 - L\right] a_{1_{b_0}}^3 + \left[\left(\frac{2\gamma n}{\gamma-1}\right)^3 - \frac{2\gamma n M}{\gamma-1}\right] a_{1_{b_0}}^2\, a_1 \\
& - \frac{2\gamma n M}{\gamma-1} a_1^2\, a_{1_{b_0}} + \left[L - \frac{2}{3}\, \frac{(3-\gamma)(2-\gamma)}{(\gamma-1)^3}\right] a_1^3 - \frac{2}{\gamma-1} a_2 \frac{u_1}{u_0} - \\
& - \frac{3-\gamma}{(\gamma-1)^2} a_1^2 \frac{u_1}{u_0} - \frac{2}{\gamma-1} a_1 \frac{u_2}{u_0} - \left(\frac{2\gamma n}{\gamma-1}\right)^2 \frac{da_{2b_0}}{d\beta} \int_{\beta-b_0}^{\beta} a_1\, d\beta' \\
& + \left[3\left(\frac{2\gamma n}{\gamma-1}\right)^3 - \frac{4\gamma n M}{\gamma-1}\right] a_{1_{b_0}} \frac{da_{1b_0}}{d\beta} \int_{\beta-b_0}^{\beta} a_1\, d\beta' - \\
& - \left(\frac{2\gamma n}{\gamma-1}\right)^2 \frac{da_{1b_0}}{d\beta} \int_{\beta-b_0}^{\beta} a_2\, d\beta' - \frac{2\gamma n M}{\gamma-1} \frac{da_{1b_0}}{d\beta} \int_{\beta-b_0}^{\beta} a_1^2\, d\beta' - \\
& - \frac{1}{2}\left(\frac{2\gamma n}{\gamma-1}\right)^3 \frac{d^2 a_{1b_0}}{d\beta^2} \left[\int_{\beta-b_0}^{\beta} a_1\, d\beta'\right]^2 - \left(\frac{2\gamma n}{\gamma-1}\right)^3 a_1 \frac{da_{1b_0}}{d\beta} \int_{\beta-b_0}^{\beta} a_1\, d\beta' \\
& + \left\{ \frac{\gamma n}{\gamma-1} (1-u_0^2) \frac{dt_{1b_0}}{d\beta} \frac{da_{1b_0}}{d\beta} - \frac{2\gamma n}{\gamma-1} \frac{da_{2b_0}}{d\beta} - \left(\frac{2\gamma n}{\gamma-1}\right)^2 \frac{d^2 a_{1b_0}}{d\beta^2} \int_{\beta-b_0}^{\beta} a_1\, d\beta' \right. \\
& - \frac{\gamma n}{\gamma-1} \frac{1-u_0}{2} \frac{d^2 a_{1b_0}}{d\beta^2} \left[t_1 - t_{1_{b_0}}\right] + \left[3\left(\frac{2\gamma n}{\gamma-1}\right)^2 - 2M\right] a_{1_{b_0}} \frac{da_{1b_0}}{d\beta} \\
& \left. + \left[\frac{4\gamma n}{(\gamma-1)^2} - \left(\frac{2\gamma n}{\gamma-1}\right)^2\right] a_1 \frac{da_{1b_0}}{d\beta} \right\} \left(\frac{1-u_0^2}{2}\right) \left[t_1 - t_{1_{b_0}}\right] - \\
& - \left(\frac{2\gamma n}{\gamma-1}\right)^2 \left(\frac{1-u_0^2}{2}\right) \left[\frac{da_{1b_0}}{d\beta} \int_{\beta-b_0}^{\beta} a_1 \frac{dt_1}{d\beta}\, d\beta' - \frac{da_{1b_0}}{d\beta} \frac{dt_{1b_0}}{d\beta} \int_{\beta-b_0}^{\beta} a_1\, d\beta'\right] \\
& - \frac{\gamma n}{\gamma-1} (1-u_0^2) \frac{da_{1b_0}}{d\beta} \left[t_2 - t_{2_{b_0}}\right]
\end{aligned}
\qquad \text{(E-17)}
$$

The last term of R_2' and the last few terms of R_3' result from expressing the retarded variable at $\beta - b_o$ rather than at β^*. It is seen that derivatives and integrals appear in this boundary condition but they always appear as nonlinear terms. So they may be calculated from lower order analysis and really are just considered as known terms in the inhomogeneous part of the higher order boundary condition. This will become clearer after the boundary condition is separated according to powers of ϵ. Then it will be seen that this condition is actually just a series of algebraic relations. (The separation procedure is being delayed until after the eigenvalue perturbation is performed in Chapter III.) These algebraic relations, however, will contain retarded variables.

In conclusion, Equation (E-17) is the nonlinear expression of the condition at the end of the combustion zone. In particular, it is the boundary condition at one chamber end for the gas-dynamic analysis of the combustion instability problem. It is based primarily on the arguments which lead to the establishment of Equations (E-1) and (E-4); i.e., it is an extension of the Crocco time-lag theory which includes nonlinear terms.